THE

MATHEMATICAL GAZETTE

EDITED BY

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THE MATHEMATICAL ASSOCIATION.

THE Annual General Meeting of the Mathematical Association was held at the University of Bristol on 28th, 29th, 30th, 31st March, 1951.

On 28th March members were welcomed by the Vice-Chancellor at a reception held in Wills Hall.

On 29th March, the business meeting was held at 9.30 a.m., with the President, Professor H. R. Hassé, in the chair. The Report of the Council for 1950 was adopted, and a financial statement given by the Treasurer was received. The following alteration of the Rules was carried: "That the following shall be added to Rule 8: 'An ordinary member who has paid fifty annual subscriptions shall be deemed to have completed his life composition without making any further payment.'"

The election of Dr. M. L. Cartwright, F.R.S., as President for 1951 was announced. The existing Vice-presidents were re-elected, and Professor Hassé was elected a Vice-president. Mr. F. W. Kellaway and Miss W. Cooke were elected Secretaries, the latter in the place of Miss M. E. Bowman, who has retired on the grounds of ill-health. The thanks of the Association were given to Miss Bowman for her loyal service to the Association. The Librarian, the Editor of the *Mathematical Gazette*, and the Auditor were re-elected. The following were elected to serve on the Council: Mr. D. G. Bousfield, Professor T. Arnold Brown, Dr. I. W. Busbridge, Mr. C. T. Daltry, Professor R. L. Goodstein, Mr. W. J. Langford, Miss F. M. Pendry, Mr. M. A. Porter, Professor A. G. Walker. Professor W. G. Bickley was elected an Honorary Member of the Association.

At 10 a.m. the Presidential address was given by Professor Hassé, the subject being "My Fifty Years of Mathematics". At 11.45 a.m., Mr. W. J. Langford spoke on "The teaching of algebra".

In the afternoon, members visited places of interest in the Bristol area, and theatre parties were arranged for the evening.

On 30th March, at 9.30 a.m., a symposium on the exhibition of work from Modern Schools was held. The organiser of the exhibition, Miss L. D. Adams, described the general plan of the exhibition and particular points were dealt with by Mr. James, Mr. North and Miss Sowden. At 11.30 a.m., Professor Powell, F.R.S., gave a lecture on "Cosmic radiation", illustrated by films and

slides. At 2.15 p.m. Mr. C. O. Tuckey and Mr. G. L. Parsons opened a discussion on the Trigonometry Report, and at 4.45 p.m. Professor H. A. Heilbron spoke on " Babylonian mathematics ". In the evening, a social gathering was held in Wills Hall.

On 31st March, at 9.30, a discussion on " Scholarship syllabuses in the General Certificate of Education and in University Examinations " was opened by Mr. P. C. Unwin, Mr. H. E. Parr, Mr. F. J. Tongue and Dr. W. L. Ferrar.

A Publishers' Exhibition was open throughout the meeting.

MATHEMATICS AT THE FESTIVAL

Throughout the period of the Festival, the College of Preceptors is holding in its premises at 2 and 3 Bloomsbury Square, London, W.C. 1, an exhibition entitled " The Teacher: In School and Out ". This exhibition, which tells the story of developments in the teaching profession during the past 100 years, will be in four sections: (1) Your Grandfather's Teacher; (2) Your Father's Teacher; (3) Your Teacher; (4) Your Child's Teacher.

The Mathematical Association is participating in these events, and, under the auspices of the London Branch, a " Mathematics Week " will be staged from Monday, 9th July, to Friday, 13th July (with the exception of the Wednesday evening). A comprehensive display will be arranged of material used in mathematical teaching in educational institutions of all kinds, and special prominence will be given to the work of the Mathematical Association since its foundation as the Association for the Improvement of Geometrical Teaching. Part of the Council Chamber will be set out as a small class-room.

The programme will be as follows:

Monday, July 9th.
- 2.30 p.m. Demonstration and discussion: Primary School (Miss Shavelson, Goldsmiths' Training College).
- 7 p.m. Lecture (Professor T. A. A. Broadbent, Royal Naval College, Greenwich).

Tuesday, July 10th.
- 2.30 p.m. Demonstration and discussion: Secondary Technical School.
- 7 p.m. Lecture (Mr. W. Cooper, Rugby College of Technology).

Wednesday, July 11th.
- 2.30 p.m. Lecture-demonstration: The mathematics class-room (Mr. G. H. Grattan-Guinness, Huddersfield Education Department).

Thursday, July 12th.
- 2.30 p.m. Demonstration and discussion: Secondary Modern School (Miss Shavelson, Goldsmiths' Training College).
- 7 p.m. Lecture (Mr. A. W. Siddons, formerly President of the Mathematical Association).

Friday, July 13th.
- 2.30 p.m. Demonstration and discussion: Secondary Grammar School (Mr. J. B. Morgan, Harrow).
- 7 p.m. Lecture (Mr. W. J. Langford, Battersea Grammar School).

Short displays (half an hour each) of films of a mathematical nature will be given at intervals during each day. Two separate programmes will be staged:

First programme: 11.30, 1.30, 5.00;
Second Programme: 12.30, 4.00, 6.00.

Mathematical filmstrips will be displayed continuously by automatic projector when lectures and demonstrations are not in progress.

REPORT OF THE COUNCIL FOR THE YEAR 1950

Membership.

During the period from 1st November, 1949, to 31st October, 1950, 128 full and 35 Junior members were admitted. The membership at 31st October, 1950, was 2,739, of whom 7 are Honorary, 228 Life Members, 2,178 Ordinary Members, 230 Junior Members and 96 Libraries. It is with regret that the Council reports the death of the following members: Mr. H. J. T. Blake (1910), Mr. J. F. Hudson (1899), Mr. W. A. Lewis (1908), Professor G. H. Livens (1930), Miss K. LeQuesne (1944), Professor J. H. M. Wedderburn (1913).

Finance.

The balance in hand on 1st November, 1949, was £279 2s. 3d., and on 1st November, 1950, was £115 1s. 11d. There was, therefore, a loss of £164 0s. 4d. during the last financial year.

Members will undoubtedly be disappointed that the increased subscription has failed to stop the succession of losses which has now covered the last five years, and has already consumed the £1,000 saved during the years of war. Fortunately, members have received some tangible evidence of the destination of that £1,000; the Trigonometry and Calculus Reports between them cost the Association £1,070 14s. 6d.

Examination of the Treasurer's statement, which was circulated with *Gazette* No. 310, shows other reasons for the unexpected loss during last year. The cost of printing is higher; so are the amounts expended on clerical assistance and stationery. These are matters which can, to some extent, be controlled by the Council through its officers and committees. But the statement does not show the considerable loss of income which was caused by some members omitting to pay the increased subscription, and by over 100 members who did not pay at all.

The outcome of our next account depends very much on members, and especially on the Branches. If we can hold our present membership, or possibly even increase it, the measures of economy which will be taken by the Council should be sufficient to stop the succession of losses. But the measures of economy—which will be designed to keep the Association functioning without loss of efficiency or delay in production of reports—cannot by themselves be sufficient; a small increase in our income from subscriptions is essential.

The Equalisation Fund now stands at £11 3s. 0d. There has been only one request for assistance—Leicester Branch received £2 11s. 0d. for the payment of lecturer's expenses. Branch Secretaries are reminded of the existence of this fund; it exists to help smaller Branches to pay the expenses of lectures which they would not otherwise be able to afford.

The Branches.

The Branches have continued to flourish throughout the year and a number of reports of their activities have been published in the *Mathematical Gazette*.

It is noteworthy that the membership of London Branch now stands at practically 500, with a fair percentage of teachers in Secondary Modern Schools.

The Mathematical Gazette.

Vol. XXXIV was issued in four parts, instead of five, and this will have to be normal practice for some years.

There is no diminution in the number of Articles and Notes submitted while there is a steady increase in the number of books sent for review.

The Teaching Committee.

The newly-elected Teaching Committee met on January 5th, 1950. A report of that meeting was given in the *Gazette* for February, 1950 (no. 307, p. 5). The two vacancies remaining in the Committee at that date have been filled by the Standing Subcommittee's nomination of Miss Eleanor Siddons and Mr. G. R. Forbes.

The Trigonometry Report was distributed to members with the July number of the *Gazette*, and it is hoped that before the Annual Meeting members will have received the Calculus Report with the *Gazette* for February, 1951. Reprints of the Algebra Report and of the First Report on the Teaching of Geometry have been ordered.

The Teaching Committee were consulted by correspondence about the Mechanics Report, and an account of the results was published in the *Gazette* for September, 1950 (no. 309, p. 198). With the co-operation of the Editor, steady use is being made of the *Gazette* as a means of obtaining publicity for questions under discussion and for telling members of the Association what is being done through the Teaching Committee.

The seven subcommittees, lettered *a* to *g* on page 7 of no. 307 of the *Gazette*, have been constantly active throughout the year. Those for Geometry, Primary Schools and Visual Aids are well on with their reports, but will not be able to submit them to the Teaching Committee before January, 1952. The Sixth Form subcommittee has split into three, dealing respectively with History of Mathematics, Sixth Form Algebra, and University Coordination (*i.e.* co-ordination of studies in the last year at school with those in the first year at the University). The Technical Colleges subcommittee has been working in two sections since 1949. A new subcommittee has been appointed to bring the Book List up-to-date. Volunteers have been sought (with some success) to check General Certificate examination papers as soon as they are set to candidates, so that appropriate comments may be passed to the examining bodies concerned.

During the year the Association received a request from the Netherlands Ministry of Education for a member to represent Great Britain at a conference in Holland on the teaching of mathematics in schools. The Association owes its thanks to Mr. W. J. Langford for going to this conference and reading a paper which provoked a lively interest in the Association's activities and publications.

The Standing Subcommittee also dealt with a request from the Civil Service Commissioners for comments on proposed changes in some of their syllabuses.

Problem Bureau.

There have been many enquiries for solutions of problems in Calculus and Analysis taken from the latest volume of Cambridge Scholarship questions; and also some from the Girton and Newnham papers. In addition, a great variety of problems have been received from other sources, more in Geometry—both Pure and Coordinate—rather less in Mechanics—than usual.

Officers and Council.

The Council wishes to express its sincere thanks to Professor H. R. Hassé for his services as President, and to the Officers for the way in which they have continued to perform their duties. Dr. F. G. Maunsell and Dr. J. Topping have retired, in rotation, from the Council and appreciation is expressed to them for their assistance.

A SCHOOL MATHEMATICAL SOCIETY.

By E. C. Witcombe.

Nearly two years ago I was given permission to form a School Mathematical Society, hoping thereby to create a live interest in mathematics and to show that there is a limitless treasure untouched by the demands of the academic courses of the School Certificates. Naturally topics have to be chosen which require little more than a knowledge of School Certificate and Higher School Certificate mathematics, and the Mathematical Society has been limited, at the present, to the Fifth and Sixth forms.

The first address was given on the Calendar, much of the material being taken from Smith's *History of Mathematics*, Sanford's *Short History of Mathematics* and *Whitaker's Almanack*.

In the hope that others may write similar articles, or form similar societies, I give a brief outline of this address.

The Calendar.

When I was preparing this talk the daily papers contained the news of the discovery of the skull of a prehistoric ape which scientists estimated to be between 20 and 25 million years old. Though not pre-man the skull showed traces of both human and animal characteristics. Did such an animal have a system of reckoning time? The question may seem at first glance to be absurd, yet the earliest known calendar, that of the Maya people who inhabited parts of Mexico and Central America, dates from the 34th century B.C. The year was divided into eighteen months and began on the shortest day.

Other early calendars include those of the Babylonians and the Chaldeans, whose scientific observations of the heavens during the first 2000 years B.C. enabled them to determine the length of the year to a high degree of accuracy. Their knowledge was not universal, so it is not surprising to find that the early Roman calendar had a year of 304 days only. The Roman calendar underwent various reforms, until finally Julius Caesar undertook a reform of the calendar in 46 B.C. (known as the Year of Confusion), adding 80 days to the current year to make the first of Spring on the calendar agree with the actual occurrence of the equinox, and for subsequent years instituting a year of 365 days with an extra day for every fourth year. This is known as the Julian or Old Style calendar.

Astronomical observations over a very considerable period of time have shown that the basic assumption of the Julian calendar of $365\frac{1}{4}$ days is incorrect. By the close of the fourteenth century the departure of Easter Day from its traditional position had become very noticeable, and in 1582 Pope Gregory XIII reformed the calendar, decreeing that 4th October, 1582, should be called 15th October, and that leap years should occur at the centuries divisible by 400, starting at 1600. This calendar is known as the Gregorian or New Style calendar; it was adopted in England in 1752 (14th September).

In addition to stabilising the Roman calendar, Julius Caesar was responsible for the year of 12 months and for the alternation of 31- and 30-day months, February excepted. Since Julius Caesar there have been efforts to alter the number of days in each month. Most notable of these include the calendar of the French Revolution, which lasted for 13 years, with its 12 months of 30 days and 5 or 6 days grouped at the end of the year as holidays, and the World Calendar now before the United Nations. The latter consists of a year of twelve months in four equal quarters. Every year begins on Sunday, 1st January; each quarter consists of three months, the first of 31 days and the other two of 30; each quarter begins on a Sunday; and holidays are stabilised with 31st December as a World's Day and a world holiday.

No matter which calendar is used, it is quite easy to devise a system by which it is possible to determine the day of the week for any particular happening of the past or future of given date. Dealing with the Gregorian or New Style calendar first, the following table gives a fixed or golden number for each month (the table can easily be extended).

New Style.

	1753–99	1800–99	1900–99	2000–99	2100–99
January	5	3	1	6	4
February	1	6	4	2	0
March	1	6	4	3	1
April	4	2	0	6	4
May	6	4	2	1	6
June	2	0	5	4	2
July	4	2	0	6	4
August	0	5	3	2	0
September	3	1	6	5	3
October	5	3	1	0	5
November	1	6	4	3	1
December	3	1	6	5	3

The rule is as follows : to the last two digits of the year, add the number of leap years ; add the date of the month and the golden figure for the month ; divide the result by 7 and the remainder is the day of the week on the scale Sunday $=1$, ..., Friday $=6$, Saturday $=0$. For example, December 25th 1949, is a Sunday, since $(49+12+25+6)$ divided by 7 leaves a remainder 1.

Interesting historical dates can similarly be fixed :

the Wright Brothers first flew in a heavier-than-air machine on 17th December, 1903, a Thursday ;
the battle of Waterloo, 18th June, 1815, a Sunday ;
the battle of Trafalgar, 21st October, 1805, a Monday ;
Napoleon Bonaparte born 15th August, 1769, a Tuesday.

The table can also be made to predict the day of the week of any coming event ; for instance, Christmas Day in the year 2000 will be a Monday.

Similarly it can be made to determine the dates of all the Saturdays in November 1950, for the total must be exactly divisible by 7, so that

$$(50+12+4+x)$$

must be divisible by 7, so that $x=4$, 11, 18 or 25. Likewise, for the Mondays of February in 1965 the remainder on division must be 2, and the dates are the 1st, 8th, 15th and 22nd. A leap year occurs after 29th February ; for example, for 12th February, 1936 the number of leap years is 8.

The corresponding table for the Julian or Old Style calendar is as follows (the table can easily be extended) :

Old Style.

	1700–52	1600–99	1500–99	1400–99	1300–99
January	2	3	4	5	6
February	5	6	0	1	2
March	5	6	0	1	2
April	1	2	3	4	5
May	3	4	5	6	0
June	6	0	1	2	3
July	1	2	3	4	5
August	4	5	6	0	1

	1750–52	1600–99	1500–99	1400–99	1300–99
September -	0	1	2	3	4
October -	2	3	4	5	6
November -	5	6	0	1	2
December -	0	1	2	3	4

	1200–99	1100–99	1000–99	900–99	800–99
January -	0	1	2	3	4
February -	3	4	5	6	0
March - -	3	4	5	6	0
April - -	6	0	1	2	3
May - -	1	2	3	4	5
June - -	4	5	6	0	1
July - -	6	0	1	2	3
August -	2	3	4	5	6
September -	5	6	0	1	2
October -	0	1	2	3	4
November -	3	4	5	6	0
December -	5	6	0	1	2

Interesting historical dates include :

Battle of Hastings, 14th October, 1066, a Saturday ;
Charles I beheaded, 30th January, 1649, a Tuesday ;
Columbus set sail, 3rd August, 1492, a Friday ;
Frederick, Prince of Wales, died from a hit on the head from a cricket ball, 20th March, 1751, a Wednesday ;
Fire of London began, 2nd September, 1666, a Sunday.

E. C. W.

GLEANINGS FAR AND NEAR.

1665. A Curiosity. $7x^4-112x^3+672x^2-1792x+1801$ is a square for $x=1, 2, 3, 4, 5, 6, 7, -8, 16$. [Per Mr. R. C. Lyness.]

1666. Omar appointed a committee of six, and ordered that in the event of four taking one side and two the other, the minority should be decapitated. His committee . . . declined to act ; and it is obvious that a few divisions on this principle would reduce even our House of Commons to *one*.—D. S. Margoliouth, *Mohammedanism*, p. 94. [Per Mr. J. W. Ashley Smith.]

1667. Ellipsoid. Solid of which all plane sections through one axis are ellipses and through the other ellipses or circles.

Paraboloid. Solid, some of whose plane sections are parabolas.—*Concise Oxford Dictionary* (1931). The hyperboloid apparently defied definition. [Per Mr. G. A. Bull.]

1668. A colleague thinks his wife has discovered the peak of the inflationary spiral. Searching the local shops for cherry sticks she found one which stocked them—at 1s. 6d. a dozen. Pre-war price, 144 for 3d. Price increase, 7,200 per cent.—*News Chronicle*, December 2, 1948. [Per Mr. C. B. Gordon.]

ON CERTAIN CONFIGURATIONS OF CONGRUENT TRIANGLES. II.

By D. G. Taylor.

1. This is a sequel to a paper with similar title (*Gazette*, Vol. XXXI, pp. 270–8), which will be referred to as *C.T.* (I). It concerns Case I (§§ 8–11). For clearness of diagrams, a new shape of triangle has been used; and, to avoid confusion with previous results, and to suit the present development, different letters and suffixes are introduced.

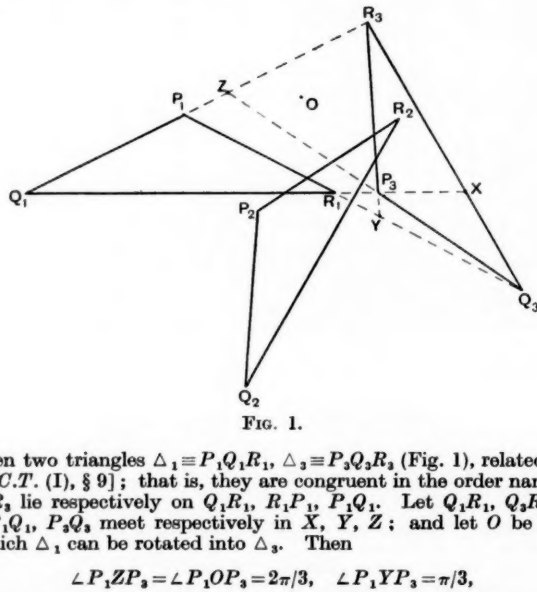


Fig. 1.

2. Given two triangles $\Delta_1 \equiv P_1Q_1R_1$, $\Delta_3 \equiv P_3Q_3R_3$ (Fig. 1), related as ABC, $A_1B_1C_1$ [*C.T.* (I), § 9]; that is, they are congruent in the order named; also P_3, Q_3, R_3 lie respectively on Q_1R_1, R_1P_1, P_1Q_1. Let Q_1R_1, Q_3R_3; R_1P_1, R_3P_3; P_1Q_1, P_3Q_3 meet respectively in X, Y, Z; and let O be the point about which Δ_1 can be rotated into Δ_3. Then

$$\angle P_1ZP_3 = \angle P_1OP_3 = 2\pi/3, \quad \angle P_1YP_3 = \pi/3,$$

and hence P_1, Z, O, P_3, Y lie on a circle. Similarly Q_1XOQ_3Z, R_1YOR_3X are circles. Let these be called *the triad of circles determined by* Δ_1, Δ_3. Let their centres be called respectively P_2, Q_2, R_2; these are the vertices of a new triangle $\Delta_2 \equiv P_2Q_2R_2$.

The triangles P_2P_1O, P_2P_3O are congruent; indeed, since the angles P_2OP_1, P_2OP_3 are each equal to $\pi/3$, they are equilateral; thus OP_2 is equal to OP_1 and OP_3, and bisects the angle between them. Similarly for OQ_2, OQ_1, OQ_3; and also for OR_2, OR_1, OR_3. Thus Δ_2 occupies the position of Δ_1 when it has rotated halfway round towards Δ_3.

The centres P_2, Q_2, R_2 of the triad of circles determined by Δ_1, Δ_3, are the vertices of a triangle Δ_2 congruent to these, namely the triangle halfway between them by rotation.

3. Now [*C.T.* (I), § 9] there is a third triangle which we may call $\Delta_5 \equiv P_5Q_5R_5$, such that Δ_1, Δ_3, Δ_5 are cyclically related, the vertices of each lying on the sides of the preceding, in the Case I order. Hence Δ_3, Δ_5 determine a triad of circles, whose centres are the vertices of a triangle Δ_4;

and $\triangle_5$, $\triangle_1$ determine a triad of circles, whose centres are the vertices of a triangle $\triangle_6$. The six triangles $\triangle_r$ ($r=1, 2, \ldots, 6$) are obtained when any one of them makes successive rotations of $\pi/3$ about O; the vertices of each are the centres of the triad of circles determined by the pair between which it lies; and the alternate triangles $\triangle_2$, $\triangle_4$, $\triangle_6$ form a trio with the same cyclical property as $\triangle_1$, $\triangle_3$, $\triangle_5$.

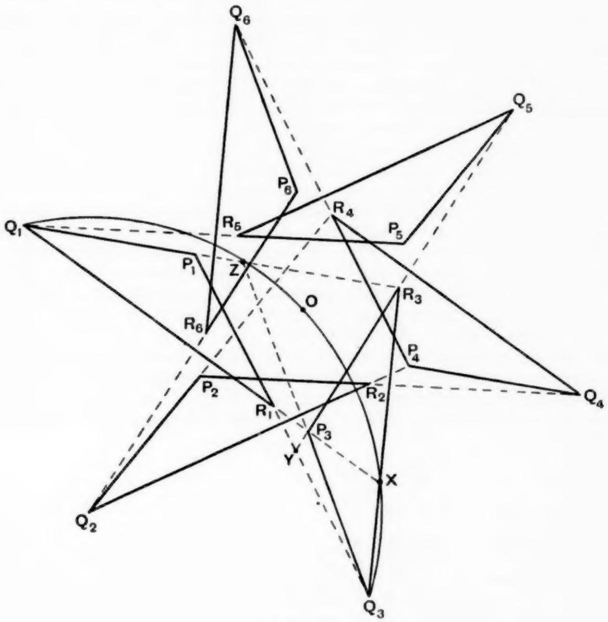


FIG. 2.

4. A representation of the six triangles is given in Fig. 2. To avoid confusion, most of the dotted lines, indicating collinearities, are omitted, but those showing the relations of $\triangle_1$ to $\triangle_3$ and of $\triangle_2$ to $\triangle_4$ are given. For the same reason, of the eighteen circles constituting the six triads, only one is shown, namely, Q_1ZOXQ_3, centre Q_2. D. G. T.

AN APPEAL.

Members who possess copies of past *Gazettes* for which they have no further use can assist the Association by returning these copies to the Editor. Back numbers can be used to make up complete volumes, for which there is a steady demand, so that their return can be of considerable financial help to the Association. Nos. 273–292 (Vols. XXVII–XXX) are particularly desirable.

THE LOGARITHMIC ABACUS.

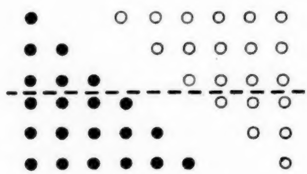
By K. F. Solloway.

"*One of the fruits of the higher education is the illuminating view that a logarithm is merely a number that is found in a table. We shall have to widen the curriculum.*" (Kasner and Newman)

As H. G. Wells pointed out a long time ago, in the country of the blind the two-eyed man tends to misjudge his neighbours. Relying almost entirely upon his power of vision, he ignores the other senses, and thinks that to be blind is to be mentally defective. In fact, a blind person sometimes has a more balanced mind than the man with keen edges, ten thumbs and ingrowing ears!

It is because this article is essentially a work of blind ignorance that it may have some pathological interest for real mathematicians. I hardly know e from a bull's foot—though I like the strange taste of π. Nursed in the Arts, corrupted by the inductive reasoning upon which lawyers are forced to rely, I shall always be mathematically blind. However, I may in time acquire sufficient logic of the baser kind to convince mathematicians that though the blind have no eyes, they can often be taught to see with their fingers. Like embossed patterns, the beauties of Number are perhaps to be fully apprehended only by a mental sense of touch—a kind of finger-sight! For this reason I distinguish from all other teaching devices those which encourage, not mental vision, but mental *grasp*. Such devices do not merely *illustrate* principles, or provide abstract analogies : they *are* principles, expressed in tangible form.

In many ways the Greeks were poor mathematicians : but because of their very blindness, their lack of elementary mathematical technique, they became past-masters in the art of finger-sight! Here is a superb example of Greek thought :



That is : the sum of the first n integers is $\frac{1}{2}n(n+1)$.

This is more than a visual aid, more than a mere mathematical proof : it is a manifestation! If I am guilty of insubordinate vehemence, it is because too many mathematicians regard finger-sight as a kind of comic relief! The Greeks were *not* mental defectives.

Having no option, I have been forced to use my mental sense of touch. It was thus that at the age of 37 I achieved for the first time an understanding of certain elementary characteristics of Number, and suddenly invented logarithms! I had then been a stranger to mathematics for 21 years : I had never seen a slide-rule and knew nothing of its purpose. Of course, when a mathematically blind person kneels down and gropes with careful hands over the groundwork of Number, he inevitably makes discoveries. I was lucky enough to find the logarithmic abacus, which I believe to be a " manifestation ".

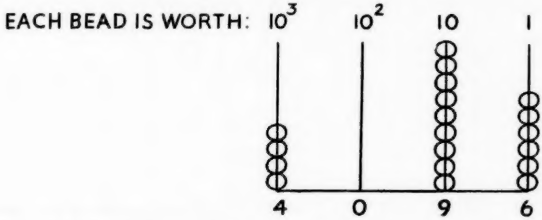
I began by appreciating the importance of Arabic notation, and by realising that digits describe the appearance of an imaginary abacus (see figure). Every time a bead is moved one wire to the left, its value is multiplied by 10.

Clearly, 10^2 means " Multiply ONE by 10 *twice* ".*

And 10^1 is 10.

And 10^0 must mean " Don't multiply ONE by 10 at all " or " Tell 10 to leave ONE alone! "

That is : $10^3 \div 10^2 = 10^{3-2} = 10^1$.



Every time a bead is moved one wire to the right, its value is *divided* by 10.

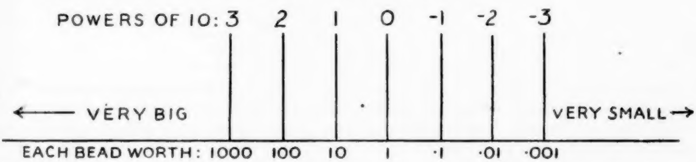
$10^3 \div 10^2$ means " Take a bead from the units wire : give it 3 pushes to the left and then 2 pushes to the right.

A principle is now established : movement to the left (or multiplication) is positive, and movement to the right (or division) is negative.

$$\tfrac{1}{10} = 10^0 \div 10^1 = 10^{0-1} = 10^{-1}.$$

Clearly 10^{-n} means " Divide ONE by 10 n times " or $10^{-n} = 1/10^n$.

Here is the Extended Abacus of 10 (Mark II) :



Even the most short-sighted person will already have seen the pretty flowers whose scent at this point distracted a blind man's attention!

Diversion : Pythagoras—blind as a bat, but an artist to his finger-tips!—was wrong in regarding ONE as the source of all numbers. Ironically enough, ONE can be regarded as a source infinitely more fruitful than he supposed, for every tangible number has its abacus, and x^0 is ONE.

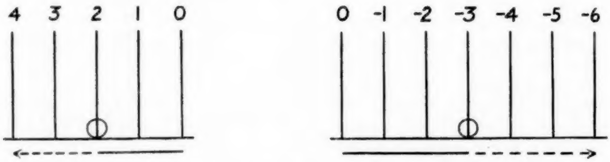
When x is	x^3	x^2	x^1	x^0	x^{-1}	x^{-2}	x^{-3}
5	125	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$	$\frac{1}{125}$
4	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$
3	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$
2	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
1	1	1	1	1	1	1	1
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$-\frac{1}{3}$	$-\frac{1}{27}$	$+\frac{1}{9}$	$-\frac{1}{3}$	1	-3	$+9$	-27

* See Note 2074 on page 281 of the December 1949 issue. I do not believe that the connection between digits, powers, indices, logarithms and functions can be *grasped* unless x^n is defined as the instruction " Multiply ONE by x n times ". Any other definition is an optical illusion!

From this elegant pattern even a blind child can deduce that every G.P. forms part of a numerical spider-web, that the first term is not merely a, but ar^0, and that the nth term is ar^{n-1}.

The last line, though numerically in keeping with the others, presents a zig-zag visual pattern instead of a sweeping curve : but the essential similarity is apparent to the sense of touch, as will be shown.

To return to the argument :

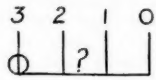


$10^2 \times 10^2 = 10^{\text{twice } 2}$, and $10^{-3} \times 10^{-3} = 10^{\text{twice } (-3)}$.

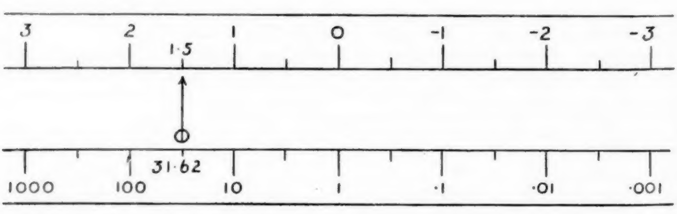
It follows that to square the value of any bead, its distance from the 0 wire must be doubled. Conversely, the operation of taking the square root is performed by halving the bead's distance from the 0 wire.

It was here that the Abacus of 10 (Mark II) revealed its limitations :

$\sqrt[2]{10^3} = 10^{1\frac{1}{2}}$.



The next moment I accidentally put my hand on something that turned out to be the Logarithmic Abacus of 10 (Mark III). I have since made a copy in balsa and cardboard : it consists of two scales with a movable wire having only one bead, thus :



Fractional indices now acquire the kind of meaning that can be *grasped* by the mathematically blind.*

$10^{3/2}$ means " 10^3 brought halfway back ", or $\sqrt[2]{10^3}$. The argument now reaches its final stage :

* I realised some time ago that the method of calculating logarithms by extracting successive square roots was already known to mathematicians, and it is ascribed by one modern text-book to Professor Perry and Mr. Edser. I am obliged to the Editor for pointing out that the method is at least as old as Briggs, and that certain comparatively recent and very authoritative German books actually define a logarithm in this way.

$$10^{\frac{1}{2}} = \sqrt{10} = 3{\cdot}162,$$

$$10^{\frac{1}{4}} = \sqrt{3{\cdot}162} = 1{\cdot}778,$$

and $$10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}} = 3{\cdot}162 \times 1{\cdot}778 = 5{\cdot}622,$$

$$10^{\frac{1}{8}} = \sqrt{1{\cdot}778} = 1{\cdot}333,$$

whence $10^{\frac{3}{8}}$, $10^{\frac{5}{8}}$ and $10^{\frac{7}{8}}$ can be calculated. By taking successive square roots we can graduate the abacus to any required degree of accuracy between 0 and 1 on the powers scale and between 1 and 10 on the Arabic scale.

Any Arabic number can now be expressed as a power of 10 :

$$1{\cdot}778 = 10^{0{\cdot}25}.$$

But $17{\cdot}78 = 10 \times 1{\cdot}778 = 10^1 \times 10^{0{\cdot}25} = 10^{1{\cdot}25}$. In every case the integral part of the index can be discovered without the aid of muttered abracadabra. For instance, 1778 is a number between 10^3 and 10^4 : it must therefore be $10^{3{\cdot}25}$. This is a *true* Rule of Thumb ; it enables the blind to handle numbers without losing their grip!

The graduated power scale from 0 to 1 runs in the left (or positive) direction. When this scale is being used in the negative part of the abacus, the movable wire must first be taken to the integral division immediately to the *right* of the required number, and then brought leftwards into position :

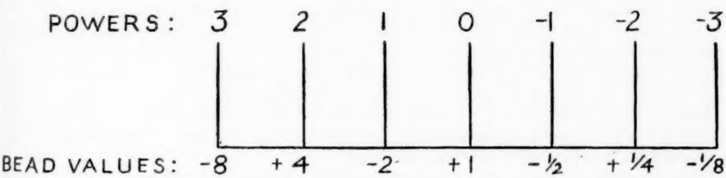
$$\cdot 01778 \text{ is a number between } \cdot 01 \text{ and } \cdot 1,$$

that is, $$\text{between } 10^{-2} \text{ and } 10^{-1},$$

$$\cdot 01778 = 10^{-2+0{\cdot}25}.$$

The Arabic numeral system is an excellent method of spelling numbers and lends itself to the processes of addition and subtraction. It permits us to "multi-ply" and "divide"—terms which exactly describe two clumsy operations! Only on the *logarithmic* abacus is it possible to "magnify" and "minimise", which are operations of a quite different kind.

The abacus of a negative number appears to oscillate Here is the abacus of -2 :



Can there be a logarithmic abacus of -2?

To find the square root of 4, the bead is "brought halfway back"—and it gives a correct answer that the logarithmic abacus of $+2$ does not supply.

To find the square root of -2 is often said to be impossible, though it is difficult to believe that $(-2)^{\frac{1}{2}}$ has no meaning. What is the locus of the moving wire between $(-2)^0$ and $(-2)^1$? The value of the bead changes from $+1$ to -2: yet it cannot pass through the impossible value NOUGHT—which no power on Earth can make! The realisation that the locus cannot be represented in two dimensions immediately causes it to take shape.

In the logarithmic abacus of -2, the Arabic scale spirals around the powers scale, upon which the movable wire must be free both to slide and to *rotate*. The readings will always correspond in magnitudes with the readings on the

logarithmic abacus of $+2$, but will correspond in *sign* at even powers only, since the sliding wire follows the spiral scale and rotates through 360° as it moves laterally through the space of two powers. Since the spiral scale is a concept relating exclusively to sign, there seems to be no good reason for assuming that it alters numerical nature.

If we now combine the logarithmic abacus of -2 with that of $+2$, we see that to perform the operation $\surd 4$, we have the choice of two different routes. When the bead is brought halfway back by the direct route it reaches $+2$, but when brought by the spiral route it reaches -2. The operation $\surd -2$ is now so indescribably simple that it can be lightly sketched with a gesture of the hand!

Since the back-and-forth symbols ($+$ and $-$) are obviously inadequate, another sign must be devised. I shall " say as I think " : and I imagine the sign to be a small square (containing an angular measurement expressed in degrees) with an attached arrow pointing in a direction conforming with the angle. This sign, which I call a gyrometer, can be simplified by omitting the square, using a North-pointing arrow for all angles between 0° and 180° and a South-pointing arrow for all angles between 180° and 360°, and omitting angular measurements at the quadrants, thus :

$(-2)^{\frac{1}{2}} = \uparrow 1{\cdot}414$ (" gyro one point four one four "),
$(-\frac{1}{3})^{\frac{3}{2}} = \downarrow 0{\cdot}192$ (" two-seventy gyro point one nine two "),
$(-1)^{\frac{1}{2}} = \uparrow 1$ (" gyro one "),
$(-1)^{\frac{3}{4}} = {}_{135}\uparrow 1$ (" one-thirty-five gyro one ").

It follows that $-1 = \leftarrow 1$, and $+1 = \rightarrow 1$.

If mathematicians now tell me that I have stuck my clumsy finger in Euler's i, I shall take their word for it and express some astonishment but very little regret. Even a blind man regards Euler with humble and affectionate respect : but his choice of the ambiguous symbol i was not a happy one. I like to believe the story that he was thwarted by printers!

I have often been told by practical mathematicians that though $\surd -1$ is an impossible or imaginary number, it has proved to be an honest operator, and is known to the trade as i. It worked, and for all they seemed to care it might have been a Sign of the Zodiac or a note of Chinese music.

Discussing the ambiguity of mathematical terms, Tobias Dantzig says :

" To the mathematician, who rarely ventures into the realm of metaphysics, these words have a very specific and quite unambiguous meaning. . . . No difficulty arises until the philosopher makes an attempt to present to the lay public his analysis of the fundamental concepts of mathematics. It is then that the different connotations attaching to such words as infinity or reality lead to hopeless confusion in the mind of the layman. This particularly applied to the concepts of *real* and *imaginary*."

This unconscious irony has a bitter flavour! Let me translate :

" No difficulty arises—unless it be at the very beginning, when the teacher makes an attempt to present to children an explanation of the fundamental concepts of mathematics. It is then. . . ."

Dantzig goes on to quote Gauss, who speaks bluntly :

" That the subject has been treated from such an erroneous point of view and enveloped with such mysterious obscurity is due largely to the inadequate terminology used."

For myself, I shall adopt the gyrometer, and its abbreviation $\uparrow$. My imagination instinctively recoils from this kind of two-way stretch :

$$i \times i \times i = i^3$$
$$= (\surd -1)(\surd -1)^2$$

$$=(\sqrt{-1})(-1)$$
$$=-\sqrt{-1}.$$

How much more eloquent is the " periphrasis " :

$$(-1)^{\frac{3}{2}}=\downarrow 1.$$

At least it is susceptible of grammatical analysis!

Even if Number is merely the language of Science—which I strenuously deny!—there can be no excuse for speaking a debased jargon in which style has no place and subject is confused with verb. In their recent book *Mathematics and the Imagination* (from which my opening quotation is also taken), Kasner and Newman say : " But the Greeks cultivated the practical only as long as it had a beautiful side ; beyond that, their mathematics was hampered by their aesthetics."

Let us take care lest our aesthetics be stultified by our mathematics! The road from Ancient Greece to Hiroshima is a Path of Progress strangely like $\leftarrow$.

At the very outset I described this article as a work of blind ignorance, and it has fulfilled its early promise : but my ignorance is my qualification. Elementary conceptions of number are derived from the abacus which therefore presents itself as one of the logical themes for a syllabus. However, the logarithmic abacus is an attempt to use the pattern of Number for educational rather than purely mathematical purposes. After all, we live in a country of the blind, and I see no reason why nineteen blind children should be driven into an intellectual wilderness merely because the twentieth child is a potential mathematician.

No doubt a chimpanzee could be trained by repetitive methods to use log tables and to solve simultaneous equations, but only Mr. Bertam Mills would think of describing the accomplished creature as an *educated* ape. It has always been a source of astonishment to me that trainers should waste their time in teaching these circus tricks to human children, whose box-office appeal (on this side of the Atlantic) cannot be compared with that of performing chimpanzees.

By what process of thought has it been decided that children shall learn *by heart* various calculating tricks connected with simple and compound interest, stocks and shares, the extraction of square roots, the solution of simultaneous and quadratic equations, and methods of simplifying and factorising improbable algebraic expressions—but that (unless their harassed teachers are prepared to risk the " waste of time ") they shall *not* know anything about topology, Arabic counterchange and other patterns, mathematical paradoxes, cycloids, the harmonograph, the theory of perspective, dissections, mazes, games like " Nim ", and amusingly instructive puzzles like " The Spider and the Fly "?

Bertrand Russell says that it is impossible to draw a line between mathematics and logic. " In fact, the two are one." If this liberal view is right, then elementary mathematics is not being taught in our schools. Those teachers who defend the present syllabus obviously recognise only one kind of logic : thus they grossly underestimate the intellectual capacity of the average child. This is shown by their insistence upon the value of technical drill, which in my opinion has a demoralising effect upon the wits. Children who are " good at Maths " usually despise commonsense and are satisfied with a solution only if it is performed by an orthodox trick : they are often completely paralysed by puzzles like " The Five Pieces of Chain ". In fact, they are taught to regard mathematics as a substitute for thought—as a systematic picking of other people's brains. Is there to be no artistry of thought?

Whenever I show my mathematical models as an entertainment at a Youth Centre, I am *invariably* asked : " Why didn't we learn this sort of mathematics at school? " Well? Why are children not taught to regard mathematics as a lifelong source of beauty and recreation?

Surely the answer is obvious : because the syllabus is designed to produce a few professionals rather than many amateurs. I see no point in mis-educating more mathematical technicians to produce more television sets to purvey more puerility to prevent the mis-educated from being bored to death with their own company : and I am simply terrified of experts who know all about atomic fission and nothing about the art of living.

Let us stop thinking that the constipating effects of the present syllabus can be purged by a dose of visual aids! We must not only " widen the curriculum " : we must consider the real scope of Elementary Mathematics, and its place in our educational system. We might even consider what purpose we intend education to serve—a subject which, if debated at a General Meeting of the Association, would provoke wide but illuminating disagreement!

K. F. SOLLOWAY.

SUMMER SCHOOL IN RELAXATION METHODS.

3rd–21st September, 1951.

In the long vacation of the years 1945–50 Summer Schools in Relaxation Methods were held at Imperial College, and in 1949–50 also in U.S.A. Their success has encouraged the provision of similar courses in both England and U.S.A. in the coming long vacation, and at Imperial College this is now planned for the three weeks 3rd–21st September, 1951.

This course will cover the numerical solution of linear algebraic equations, framework problems, Laplace's and Poisson's equations, the biharmonic equation, eigenvalue problems, the heat-conduction equation, etc.

The course will consist of daily lectures in the mornings beginning at 10.15 a.m. with numerous practical examples to be solved under supervision in the afternoons.

The fee for the course will be £10, payable to the Imperial College. The College will endeavour to provide accommodation in its Hostel building; but it may prove necessary to restrict the number so accommodated. The daily charge for a room is 9s. Meals are obtainable in the College Refectory.

Replies to this circular should be addressed to D. N. de G. Allen, Imperial College, London, S.W. 7. Separate application should be made by each individual, stating (1) the amount of time that can be given to the course, (2) whether accommodation is desired, and if so (3) whether for the whole period or for that period excluding week-ends.

THE EXTRACTION OF SQUARE ROOTS.

BY J. B. S. HALDANE.

THE method usually employed for the extraction of square roots involves a series of successive approximations, one for each significant decimal figure of the root. A binomial series may be found which converges quicker than $\Sigma 10^{-n}$, but this involves a number of separate divisions. Iteration by dividing an approximate value of the root into its square and taking the mean nearly doubles the number of significant figures at each operation. The methods here given converge rather faster than this type of iteration, and enables a root to be expressed as a simple infinite series. They arose from a question put to me by Mr. K. A. Kermack, as to how an ancient Roman could have obtained a good approximation to a square root as a rational fraction. I believe them to be novel, at least in part. It will clearly be sufficient to give methods for finding the square root of a positive integer k.

The most powerful method involves the expression of $\sqrt{k}$ as a recurrent simple continued fraction. Let P_0/Q_0 be the penultimate convergent of its first period.

Then $$P_0^2 - kQ_0^2 = \pm 1.$$

Now, let $$(P_0 - \sqrt{k} \,.\, Q_0)^{2^n} = P_n - \sqrt{k} \,.\, Q_n.$$

Then $$P_{n+1} - \sqrt{k} \,.\, Q_{n+0} = (P_n - \sqrt{k} \,.\, Q_n)^2,$$

$$P_{n+1} = P_n^2 + kQ_n^2, \qquad Q_{n+1} = 2P_nQ_n.$$

But $$P_n^2 - kQ_n^2 = (P_0^2 - kQ_0^2)^{2n} = 1.$$

So, for $n > 0$,

$$\left.\begin{aligned} P_{n+1} &= 2P_n^2 - 1 \\ Q_{n+1} &= 2^{n+1}P_nP_{n+1} \ldots P_1P_0Q_0 \end{aligned}\right\} \qquad \text{(1)}$$

Also $$\frac{P_{n+1}}{P_{n+1}} = \frac{P_n}{Q_n} - \frac{1}{Q_{n+1}},$$

$$\frac{P_n}{Q_n} = \sqrt{k} - \frac{(P_0 - \sqrt{k}Q_0)^{2^n}}{Q_n},$$

so P_n/Q_n tends to $\sqrt{k}$ as a limit, and when P_n and Q_n are large, exceeds it by $1/Q_{n+1}$, very nearly.

We may thus take P_n/Q_n as our approximation, with the above error.

Or we may put

$$\sqrt{k} = \frac{P_0}{Q_0} \pm \frac{1}{Q_1} - \frac{1}{Q_2} - \frac{1}{Q_3} - \frac{1}{Q_4} \ldots \qquad \text{(2)}$$

$$= \frac{P_0}{Q_0} \pm \frac{1}{2P_0Q_0} + \frac{1}{4P_0P_1Q_0}\left[\frac{1}{1+2P_2}\left(1 + \frac{1}{2P_3}\left(1 + \frac{1}{2P_4}\left(\ldots\ldots\right)\right)\right)\right].$$

The sign of the second term in (2) is that of $P_0^2 - kQ_0^2$. The series (2) converges with great rapidity, each term being less than the square of its predecessor. It can readily be seen that if p_n/q_n is the n-th convergent of $\sqrt{k}$ as a simple continued fraction, and t its period, then $P_n = p_{2^n t}$, $Q_n = q_{2^n t}$. This is pointed out by Chrystal (1906, p. 469) who however uses a rather roundabout process to derive P_n/Q_n.

For example, if $k = 137$,

$$\sqrt{k} = 11 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{22+}.$$

Hence $P_0 = 1744, \quad Q_0 = 149, \quad P_0^2 - kQ_0^2 = -1.$

$$P_1 = 6{,}083{,}071\ ; \quad P_0 = 74{,}007{,}505{,}582{,}081\ ;$$

$$Q_2 = 4 \times 149 \times 1{,}744 \times 6{,}083{,}071\ ;$$

$$\frac{P_2}{Q_2} = \frac{74{,}007{,}505{,}582{,}081}{6{,}322{,}889{,}991{,}014}.$$

The next term in the series is

$$\frac{-1}{2 \times 7{\cdot}40 \times 6{\cdot}32 \times 10^{25}}$$

or $1{\cdot}046 \times 10^{-27}$, approximately. Thus P_2/Q_2 is correct to 26 decimal places, P_3/Q_3 to about 52, and so on.

The process of expressing $\sqrt{k}$ as a continued fraction may however be rather tedious. Unless P_1 and Q_1 can be found readily, the following method is likely to be quicker unless 40 or more decimal places are required.

Let R_0 be the integer nearest to $\sqrt{k}$, and let $k = R_0^2 + c$, where c is a positive or negative integer.

Let $$R_n - \sqrt{k}S_n = (R_0 - \sqrt{k})^{2^n}.$$

Then $$R_{n+1} - \sqrt{k}S_{n+1} = (R_n - kS_n)^2,$$

so $$R_{n+1} = R_n^2 + kS_n^2, \qquad S_{n+1} = 2R_nS_n.$$

But $$R_n^2 - kS_n^2 = (R_1^2 - k)^{2^n} = c^{2^n}.$$

so $$\left.\begin{array}{l} R_{n+1} = 2R_n^2 - c^{2^n} \\ S_{n+1} = 2^{n+2} R_n R_{n-0} \ldots R_0 R_0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{R_{n+1}}{S_{n+1}} = \frac{R_n}{S_n} - \frac{c^{2^n}}{S_{n+1}},$$

which gives the approximate error of R_n/S_n. For example, if $k = 137$ as above, $R_0 = 12$, $c = -7$,

$$R_1 = 2.12^2 - 7 = 281\ ; \qquad R_2 = 2.281^2 - 7^2 = 157{,}873\ ;$$

$$R_3 = 2.157{,}873^2 - 7^4 = 49{,}846{,}763{,}857\ ;$$

$$S_3 = 2^3 \times 12 \times 281 \times 157{,}873\ ;$$

$$\frac{R_3}{S_3} = \frac{49{,}846{,}763{,}857}{4{,}267{,}822{,}048},$$ which is in defect by approximately

$$\frac{2401^2}{4{\cdot}585 \times 4{\cdot}268 \times 1019} \quad \text{or} \quad 3{\cdot}00 \times 10^{-14}.$$

Thus, for twenty place accuracy we should have to calculate R_4/S_4.

We can also use a method based on the repeated squaring of $p - q\sqrt{k}$, where p/q is any rational approximation to $\sqrt{k}$, including the earlier convergents.

The methods here given are probably now of no practical importance. Had they been discovered in the 17th century, as they might have been, they would have saved a good deal of computation. Even now I think if a root were wanted accurate to a hundred or more decimal places, or to several hundred binary places, they might be worthy of consideration.

J. B. S. H.

REFERENCE.

Chrystal (1906), *Algebra*, vol. 2 (London).

AN INTRODUCTION TO THE THEORY OF CONTINUOUS GROUPS.

BY R. L. GOODSTEIN.

ALTHOUGH the theory of groups has not proved itself the universal solvent Felix Klein supposed it would become, the group concept is fundamental in algebra and geometry and plays a steadily increasing part in modern mathematics. In recognition of this fact the University of Oxford (perhaps following the lead of her youngest neighbour) has recently (1949) added the theory of groups to the schedule of compulsory subjects for the Final Degree Examination in Mathematics.

The present introduction is in no respect an original contribution to the development of the subject; everything that it contains can be found in the extensive literature of continuous groups and such elements of novelty as I believe lie in the treatment are of little technical importance. My excuse for writing this note is that some beginners have found in its presentation a simpler approach than that of the standard English and American treatises.

1. *A one-parameter continuous group.*

For each value of a continuous variable a (in some interval $\mathfrak{I}$) an analytic function $\phi(x, a)$ determines a transformation of the point x to the point $x' = \phi(x, a)$. If there is a value of a, say a_0 (lying in $\mathfrak{I}$) such that $\phi(x, a_0) = x$, and if corresponding to any a, b in $\mathfrak{I}$ there is a $\lambda(a, b)$, also in $\mathfrak{I}$, such that the successive transformations

$$x' = \phi(x, a), \qquad x'' = \phi(x', b)$$

are equivalent to the single transformation $x'' = \phi(x, \lambda(a, b))$, that is to say, if $\phi(x, a)$ satisfies the functional equation

$$\phi(\phi(x, a), b) = \phi(x, \lambda(a, b)), \qquad \text{(1.1)}$$

then the transformations $x' = \phi(x, a)$ are said to form a *one-parameter continuous group*, and $\phi(x, a)$ is called a one-parameter group function. (It can be shown that all continuous solutions of the functional equation (1.1) are necessarily analytic, but here we shall suppose that $\phi(x, a)$ is analytic by definition.)

THEOREM 1. *The group property is preserved under a continuous transformation of the parameter.*

Let $\phi(x, a)$ be a group function for the range of values $m \leqslant a \leqslant M$ of the parameter, and let $a = a_0$ give the identity function. Further, let $p(t)$ be a continuous function with bounds m, M in some interval i; we have to show that $\phi(x, p(t))$ is a group with parameter t. Since a_0 lies in (m, M), by continuity there is a (least) t_0 in i such that $p(t_0) = a_0$, and therefore $\phi(x, p(t_0)) = x$. Moreover, if t and u lie in i, then $\lambda(p(t), p(u))$ lies in (m, M) and is therefore a value of the continuous function $p(w)$, attained at some point $w = v$, say, in the interval i, so that

$$\phi(\phi(x, p(t)), p(u)) = \phi(x, p(v)),$$

which proves that $\phi(x, p(t))$ is a group.

2. *The basic form for a continuous group.*

If $a_0 = 0$ and $\lambda(a, b) = a + b$, we say that the group function $\phi(x, a)$ is in *basic form.*

THEOREM 2. *Every one-parameter group function may be reduced to basic form by a differentiable transformation of the parameter.*

We have to show that if $\phi(x, a)$ is a group function we can determine $p(t)$ so that $\phi(x, p(t))$ is in basic form. By differentiating the identity

$$\phi(x', b) = \phi(x, \lambda(a, b)),$$

where $x' = \phi(x, a)$, with respect to a and b in turn, we find

$$\phi_x(x', b)\frac{\partial x'}{\partial a} = \phi_a(x, \lambda(a, b))\frac{\partial \lambda}{\partial a},$$

$$\phi_a(x', b) = \phi_a(x, \lambda(a, b))\frac{\partial \lambda}{\partial b};$$

whence, writing

$$\Omega(a, b) = \frac{\partial \lambda}{\partial a}\Big/\frac{\partial \lambda}{\partial b}$$

and $\qquad \Xi(x, b) = \phi_a(x, b)/\phi_x(x, b),$

we have

$$\frac{\partial x'}{\partial a} = \Omega(a, b)\ \Xi(x', b). \qquad \text{.....................(2.1)}$$

Since a, b are independent variables and x' is a function of x and a only, it follows that the right-hand side of equation (2.1) is independent of b. Writing

$$\omega(a) = \Omega(a, a_0), \quad \xi(x) = \Xi(x, a_0),$$

we find

$$\frac{\partial x'}{\partial a} = \omega(a)\xi(x').$$

Let $t = \int_{a_0}^{a} \omega(a)\, da$, so that t vanishes for $a = a_0$ and

$$\frac{\partial x'}{\partial t} = \frac{\partial x'}{\partial a}\Big/\frac{\partial t}{\partial a} = \xi(x'),$$

and therefore

$$\int_x^{x'} \frac{du}{\xi(u)} = t,$$

or writing

$$V(u) = \int du/\xi(u), \qquad V(x') = V(x) + t.$$

In any interval in which $\xi(u)$ is of constant sign and different from zero, $V(u)$ has a unique differentiable inverse $V^{-1}(u)$, and so

$$x' = V^{-1}(V(x) + t). \qquad \text{.................................(2.2)}$$

Accordingly, if $x'' = V^{-1}(V(x') + u)$, then $x'' = V^{-1}(V(x) + t + u)$ so that equation (2.2) expresses the group in basic form.

It follows from Theorem 2 that the transformations of a continuous group are *reversible*. For if $x' = \phi(x, a)$ is the equation of the group in basic form, then

$$\phi(x', -a) = \phi(\phi(x, a), -a) = \phi(x, 0) = x. \qquad \text{.................(2.3)}$$

3. *The key of a one-parameter group.*

If $\phi(x, a)$ is a continuous group we call $\phi_a(x, a_0)$ the *key* (or *infinitesimal transformation*) of the group. We observe that a differentiable transformation of the parameter simply multiplies the key by a constant factor, for

$$\frac{\partial}{\partial t}\phi(x, p(t)) = \phi_a(x, p(t))\, p'(t)$$

and so

$$\phi_t(x, p(t_0)) = \phi_a(x, a_0)\, p'(t_0),$$

where t_0 is given by Theorem 1. Accordingly, the key of a group contains an arbitrary multiplicative factor.

THEOREM 3. *A one-parameter group is completely determined by its key.*

Let $k\xi(a)$ be the given key of a continuous group $\phi(x, a)$, where k is an arbitrary constant; by Theorem 2 we may, without loss of generality, suppose that $\phi(x, a)$ is in basic form. Then $\phi_a(x, 0) = k\xi(x)$ and so, for any x,

$$\{\phi(x, \alpha) - \phi(x, 0)\}/\alpha \to k\xi(x) \quad \text{as } \alpha \to 0;$$

in particular, if $x' = \phi(x, a)$,

$$\{\phi(x', \alpha) - \phi(x', 0)\}/\alpha \to k\xi(x') \quad \text{as } \alpha \to 0.$$

Since $\phi(x, a)$ is in basic form, $\phi(x', 0) = \phi(x, a)$ and $\phi(x', \alpha) = \phi(x, a + \alpha)$; therefore

$$\frac{\partial x'}{\partial a} = \lim_{\alpha \to 0} \{\phi(x, a + \alpha) - \phi(x, a)\}/\alpha = k\xi(x'),$$

whence

$$x' = V^{-1}\{V(x) + ak\},$$

which is the equation of the group in basic form. Of course, the product ak is just a single effective parameter and may be replaced by a.

The equation of a group in terms of its key admits a very elegant expression in a Taylor series. Writing

$$U \text{ for the operator } \xi(x)\frac{d}{dx} \quad \text{and} \quad U' \text{ for } \xi(x')\frac{d}{dx'},$$

we have for any function $f(x')$,

$$\Big[U'f(x')\Big]_{a=0} = \xi(x)\frac{d}{dx}f(x) = U\Big[f(x')\Big]_{a=0},$$

and so

$$\Big[U'^{n+1}f(x')\Big]_{a=0} = \Big[U'\{U'^n f(x')\}\Big]_{a=0} = U\Big[U'^n f(x')\Big]_{a=0},$$

whence, by induction, $\Big[U'^n f(x')\Big]_{a=0} = U^n f(x)$ for all n. Furthermore,

$$\frac{\partial f(x')}{\partial a} = \frac{\partial x'}{\partial a}\,\frac{df(x')}{dx'} = kU'f(x'),$$

so that

$$\partial^n f(x')/\partial a^n = k^n U'^n f(x'),$$

whence by Taylor's expansion,

$$x' = \sum_{n \geq 0} \frac{a^n}{n!} k^n \Big[U'^n x'\Big]_{a=0} = \sum_{n \geq 0} \frac{(ak)^n}{n!} U^n x.$$

Theorem 3 may be expressed by saying that *a group function is completely determined by the knowledge of its partial derivative on a particular line.*

4. *A two-parameter continuous group.*

The extension of the theory to functions of more than one variable introduces no new idea, and we turn now to groups of two parameters.

The transformations $x' = \phi(x, a, b)$ form a group of two parameters if there are constants a_0, b_0 and functions $\lambda = \lambda(a, b, a', b')$, $\mu = \mu(a, b, a', b')$ such that

$$\phi(x, a_0, b_0) = x,$$
$$\phi(\phi(x, a, b), a', b') = \phi(x, \lambda, \mu).$$

The key of the group is

$$k_1\phi_a(x, a_0, b_0) + k_2\phi_b(x, a_0, b_0),$$

where k_1, k_2 are arbitrary constants.

Writing $x' = \phi(x, a, b)$ and differentiating the identity

$$\phi(x', a', b') = \phi(x, \lambda, \mu)$$

with respect to the independent variables a, b, a', b', we find

$$\phi_{x'}\frac{\partial x'}{\partial a} = \phi_\lambda\frac{\partial\lambda}{\partial a} + \phi_\mu\frac{\partial\mu}{\partial a}, \quad \phi_{x'}\frac{\partial x'}{\partial b} = \phi_\lambda\frac{\partial\lambda}{\partial b} + \phi_\mu\frac{\partial\mu}{\partial b},$$

$$\phi_{a'} = \phi_\lambda\frac{\partial\lambda}{\partial a'} + \phi_\mu\frac{\partial\mu}{\partial a'}, \qquad \phi_{b'} = \phi_\lambda\frac{\partial\lambda}{\partial b'} + \phi_\mu\frac{\partial\mu}{\partial b'},$$

whence, writing

$$\Xi_1(x', a', b') = \phi_a/\phi_{x'}, \quad \Xi_2(x', a', b') = \phi_{b'}/\phi_{x'}$$

and

$$\frac{\Omega_{11}(a, b, a', b')}{\partial(\lambda, \mu)/\partial(a, b')} = \frac{\Omega_{12}}{\partial(\lambda, \mu)/\partial(a', a)} = \frac{\Omega_{21}}{\partial(\lambda, \mu)/\partial(b, b')} = \frac{\Omega_{22}}{\partial(\lambda, \mu)/\partial(a', b)}$$

$$= \frac{1}{\partial(\lambda, \mu)/\partial(a', b')},$$

we find

$$\frac{\partial x'}{\partial a} = \Omega_{11}\Xi_1 + \Omega_{12}\Xi_2,$$

$$\frac{\partial x'}{\partial b} = \Omega_{21}\Xi_1 + \Omega_{22}\Xi_2.$$

Since x' is a function of x, a and b only, the right hand sides of these equations are independent of a', b', therefore, if

$$\Omega_{r,s}(a, b, a_0, b_0) = \omega_{r,s}(a, b), \quad \Xi_r(x', a_0, b_0) = \xi_r(x'),$$

$$\frac{\partial x'}{\partial a} = \omega_{11}\xi_1(x') + \omega_{12}\xi_2(x'),$$

$$\frac{\partial x'}{\partial b} = \omega_{21}\xi_1(x') + \omega_{22}\xi_2(x').$$

These equations are known as the first fundamental equations of the theory of continuous groups.

Let e_1, e_2 be two arbitrary constants, then in any region in which the determinants

$$\begin{vmatrix} \omega_{11} & \omega_{12} \\ e_1 & e_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} e_1 & e_2 \\ \omega_{21} & \omega_{22} \end{vmatrix}$$

do not vanish, the differential equation

$$\frac{da}{\begin{vmatrix} e_1 & e_2 \\ \omega_{21} & \omega_{22} \end{vmatrix}} = \frac{db}{\begin{vmatrix} \omega_{11} & \omega_{12} \\ e_1 & e_2 \end{vmatrix}},$$

with the initial condition $a = a_0$, $b = b_0$, determines a and b each as an analytic function of the other.

Accordingly, where the ratios

$$\begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix} \Big/ \begin{vmatrix} e_1 & e_2 \\ \omega_{21} & \omega_{22} \end{vmatrix}, \quad \begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix} \Big/ \begin{vmatrix} \omega_{11} & \omega_{12} \\ e_1 & e_2 \end{vmatrix}$$

are continuous and of constant sign, the differential equations

$$\frac{da}{\begin{vmatrix} e_1 & e_2 \\ \omega_{21} & \omega_{22} \end{vmatrix}} = \frac{db}{\begin{vmatrix} \omega_{11} & \omega_{12} \\ e_1 & e_2 \end{vmatrix}} = \frac{dt}{\begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix}},$$

with the initial conditions $a=a_0$, $b=b_0$, $t=0$, determine a and b as analytic functions of t which satisfy the equations

$$\omega_{11}\frac{\partial a}{\partial t} + \omega_{21}\frac{\partial b}{\partial t} = e_1, \quad \omega_{12}\frac{\partial a}{\partial t} + \omega_{22}\frac{\partial b}{\partial t} = e_2,$$

so that

$$\frac{\partial x'}{\partial t} = e_1\xi_1(x') + e_2\xi_2(x').$$

Writing

$$V(u) = \int du/\{e_1\xi_1(u) + e_2\xi_2(u)\},$$

we have

$$V(x') = V(x) + t,$$

and so (where $e_1\xi_1(x) + e_2\xi_2(x)$ does not vanish)

THEOREM 4. *A two-parameter group may be reduced to the basic form*

$$x' = V^{-1}\{V(x) + t\}.$$

There are only two effective parameters in this representation, for instance, e_1t and e_2/e_1.

5. Taking $a=a_0$, $b=b_0$ in the fundamental equations, we find

$$\phi_a(x, a_0, b_0) = \omega_{11}(a_0, b_0)\,\xi_1(x) + \omega_{21}(a_0, b_0)\,\xi_2(x),$$
$$\phi_b(x, a_0, b_0) = \omega_{21}(a_0, b_0)\,\xi_1(x) + \omega_{22}(a_0, b_0)\,\xi_2(x),$$

and therefore

$$e_1\xi_1(x) + e_2\xi_2(x) = k_1\phi_a(x, a_0, b_0) + k_2\phi_b(x, a_0, b_0),$$

provided that k_1, k_2 satisfy

$$k_1\omega_{11}(a_0, b_0) + k_2\omega_{21}(a_0, b_0) = e_1, \quad k_1\omega_{12}(a_0, b_0) + k_2\omega_{22}(a_0, b_0) = e_2.$$

Since, by Theorem 4, the group is completely determined by the function $e_1\xi_1(x) + e_2\xi_2(x)$, where e_1, e_2 are arbitrary constants, it follows that

THEOREM 5. *A two parameter group is completely determined by its key.*

6. *An example on groups in two and three dimensions.*

Regarding x as a position vector in n-dimensional space, the functional equation 1.1. defines a one-parameter group in n dimensions. Accordingly, the equations

$$x_r' = \phi_r(x_1, x_2, x_3, a), \quad r = 1, 2, 3$$

determine a three-dimensional group if

$$x_r = \phi_r(x_1, x_2, x_3, a_0)$$

and

$$\phi_r(\phi_1, \phi_2, \phi_3, b) = \phi_r(x_1, x_2, x_3, \lambda(a, b)\,), \quad r = 1, 2, 3.$$

We shall prove that if $x' = \phi(x, y, a)$, $y' = \psi(x, y, a)$ are the equations of a group then

$$x' = \phi(x, y, a), \quad y' = \psi(x, y, a), \quad z' = (\psi_x + z\psi_y)/(\phi_x + z\phi_y)$$

is a three-dimensional group.

From the identities $\phi(x, y, a_0)=x$, $\psi(x, y, a_0)=y$, it follows that

$$\phi_x(x, y, a_0)=1, \quad \phi_y(x, y, a_0)=0, \quad \psi_x(x, y, a_0)=0, \quad \psi_y(x, y, a_0)=1$$

and from

$$\phi(x, y, \lambda(a, b))=\phi(x', y', b), \qquad \psi(x, y, \lambda(a, b))=\psi(x', y', b),$$

we find

$$\phi_x(x, y, \lambda)=\phi_{x'}(x', y', b)\,.\,\phi_x+\phi_{y'}\psi_x, \qquad \phi_y(x, y, \lambda)=\phi_{x'}\phi_y+\phi_{y'}\psi_y,$$

$$\psi_x(x, y, \lambda)=\psi_{x'}(x', y', b)\,.\,\phi_x+\psi_{y'}\psi_x, \qquad \psi_y(x, y, \lambda)=\psi_{x'}\phi_y+\psi_{y'}\psi_y,$$

whence

$$\frac{\psi_x(x, y, \lambda)+z\psi_y(x, y, \lambda)}{\phi_x(x, y, \lambda)+z\phi_y(x, y, \lambda)}=\frac{\psi_{x'}(x', y', b)+z'\psi_{y'}(x', y', b)}{\phi_{x'}(x', y', b)+z'\phi_{y'}(x', y', b)}$$

which completes the proof.

If $\xi(x, y)=\phi_a(x, y, a_0)$, $\eta(x, y)=\psi_a(x, y, a_0)$, then the key of the group $x'=\phi(x, y, a)$, $y'=\psi(x, y, a)$ is the pair of functions $k\xi(x, y)$, $k\eta(x, y)$; to find the key of the three-dimensional group it remains only to evaluate $(\partial z'/\partial a)$ at $a=a_0$. We have

$$\left\{\frac{\partial}{\partial a}(\phi_x+z\phi_y)\right\}_{a=a_0}=\frac{\partial}{\partial x}\phi_a(x, y, a_0)+z\frac{\partial}{\partial y}\phi_a(x, y, a_0)=\xi_x+z\xi_y,$$

and similarly

$$\left\{\frac{\partial}{\partial a}(\psi_x+z\psi_y)\right\}_{a=a_0}=\eta_x+z\eta_y,$$

and so

$$\left(\frac{\partial z'}{\partial a}\right)_{a=a_0}=\eta_x+z\eta_y-z(\xi_x+z\xi_y).$$

R. L. G.

1669. Meriden Academy.—Young Gentlemen are genteelly Boarded, and carefully instructed in the English Grammar, Writing (in all the useful Hands), Arithmetic (Vulgar and Decimal), Algebra, Book-keeping, after the Italian Method of Double-entry, Geometry, Trigonometry, Surveying Gauging and Mensuration, in all its various Parts, Dialing, Geography, and a Use of the Globes, Fluxions, Astronomy and Navigation, etc.

BY R. FOSTER.

Also, The polite Accomplishments of French, Drawing and Dancing. Every necessary attention is paid to their Morals and Recreation.—*The Coventry Standard*, March 31, 1777. Were "fluxions" widely taught in provincial schools at this time? Meriden is a village about six miles from Coventry on the Birmingham road, but the academy is long since extinct. R. Foster is not mentioned in *D.N.B.* [Per Mr. C. C. Puckette.]

1670. Casual Labour. It is impossible, she [Olive Schreiner] says, to command the brain to produce artistic creations, as you can casually employ it to solve a mathematical problem.—Vera Buchanan-Gould, *The Life and Writings of Olive Schreiner*. [Per Professor A. G. Walker.]

1671. He blew out a puff of smoke with the air of a professor of mathematics dotting the final full-stop to a thesis proving the inadmissibility of Vandermonde's Theorem.—James Curtis, *The Gilt Kid* (Penguin edition), p. 87. [Per Mr. R. O. Davies.]

ON THE INTERSECTIONS OF A CENTRAL CONIC AND ITS PRINCIPAL HYPERBOLAS.

By C. T. Rajagopal.

1. Let an ellipse or a hyperbola be given by the equation

$$S \equiv \frac{x^2}{a^2} \pm \frac{y^2}{b^2} - 1 = 0. \quad \text{.................................(1)}$$

Then, following Vaidyanathaswamy [8, p. 124],* we shall refer to a hyperbola H having its asymptotes parallel to the principal axes of S as a principal hyperbola of S. H is thus given by an equation of the type

$$H \equiv 2xy + 2gx + 2fy + c = 0, \quad \text{..............................(2)}$$

and it is uniquely determined when we know three of its intersections with S. It is one of the purposes of this article to bring together some of the elementary properties of H which have been ignored, quite unjustifiably, by most of the textbooks on conic sections produced after Casey's *Analytical Geometry* [2]. Another purpose which it is hoped the article will serve is that of bringing within easy reach of students, who have had only a preliminary course in geometries of two and three dimensions, some of Vaidyanathaswamy's investigations into the concurrency of normals to a central conic [6].

The theory of normals to S is bound up with the properties of the hyperbolas H for which $c=0$, while a consideration of the more general hyperbolas H for which $c \neq 0$ leads to the theory of θ-normals. The θ-normal at a given point of any curve is the straight line through the point which makes an angle θ with the tangent at the point to the curve, so that, taking the positive direction of rotation to be the direction from the positive x-axis to the positive y-axis (*i.e.* counter-clockwise), we have the equation of the θ-normal at (x_1, y_1) to S:

$$x_1 y_1 \left(\frac{1}{a^2} \mp \frac{1}{b^2}\right) - \frac{x_1}{a^2}(x \cot\theta + y) \pm \frac{y_1}{b^2}(x - y\cot\theta) + \left(\frac{x_1^2}{a^2} \pm \frac{y_1^2}{b^2}\right)\cot\theta = 0,$$

or, since $x_1^2/a^2 \pm y_1^2/b^2 = 1$,

$$x_1 y_1 \left(\frac{1}{a^2} \pm \frac{1}{b^2}\right) - \frac{x_1}{a^2}(x \cot\theta + y) \pm \frac{y_1}{b^2}(x - y\cot\theta) + \cot\theta = 0.$$

The substitutions $x = \xi$, $y = \eta$ in this equation yield the condition for the θ-normal to S at (x_1, y_1) to pass through (ξ, η); and, when we drop the suffix 1 attached to x and y in the condition, we obtain the equation of the curve on which the feet of the θ-normals from (ξ, η) to S lie in the form

$$T \equiv \left(\frac{1}{a^2} \pm \frac{1}{b^2}\right) xy - (\xi \cot\theta + \eta)\frac{x}{a^2} \pm (\xi - \eta\cot\theta)\frac{y}{b^2} + \cot\theta = 0. \quad \text{......(3)}$$

Therefore there are four θ-normals from any point (ξ, η) to S and their feet lie on the conic T which is called a Tesch hyperbola [2, p. 538].† The intersections of T and S which are the feet of the θ-normals in question may be called, following the usual practice [7, p. 281] a quasi-normal or quasi-pedal tetrad of deviation θ. It is thus clear that the Tesch hyperbolas (3) corresponding to a given deviation θ form a net or linear two-parameter system of conics,

* Throughout this paper, numbers in heavy type within square brackets refer to the literature cited at the end; also axes of co-ordinates are rectangular.

† It may be noted incidentally that when (ξ, η) describes any curve in the plane of S, the centre of T describes another coplanar curve of the same degree which passes through the centre of S if and only if the first curve passes through the centre of S.

while the totality of Tesch hyperbolas for all deviations θ constitute the principal hyperbolas (2). In fact, a given principal hyperbola may be identified with a Tesch hyperbola which cuts out on S a quasi-normal tetrad of deviation θ determined by

$$\cot\theta = \frac{c}{2}\left(\frac{1}{a^2} \pm \frac{1}{b^2}\right).$$

In the particular case in which $\theta = \pi/2$ or $c = 0$, and only in that case, the principal hyperbola H cuts out on S a normal or pedal tetrad and reduces to an Apollonian hyperbola.

2. There is a simple geometrical specification of any three of the intersections of S and H when H has become an Apollonian hyperbola. This is a specification of what (in our terminology) may be called a pedal triad of points on S; and, following Vaidyanathaswamy [**6**, p. 302], we may state it as under.

Theorem 1. *One of the following conditions is necessary and sufficient for three points on S to make a pedal triad :*

(*a*) *the triangle formed by the three points is self-polar w.r.t.* a conic coaxal with S ;*

(*b*) *the triangle formed by the three points circumscribes a parabola touching the axes of S.*

Vaidyanathaswamy [**6**, pp. 300–2] proves the above theorem from certain apolarity considerations. But one can also present the proof in more familiar terms, appealing to the two lemmas stated below of which the first has been noticed already and the other is well-known.

Lemma 1. *A necessary and sufficient condition for three points on S to be a pedal triad is that the principal hyperbola H determined by the three points should pass through the origin.*

Lemma 2. *Any one of the following relations between two triangles implies the other two :*

(i) *Two triangles are both inscribed in the same conic H.*
(ii) *Two triangles are both self-polar w.r.t. the same conic H′.*
(iii) *Two triangles are both circumscribed to the same conic H″.*

Proof of Theorem 1. Let α, β, γ be three points on S. Let o be the centre of S and let the x- and y-axes meet the line at infinity in ω_1 and ω_2.

If α, β, γ form a pedal triad, then (by Lemma 1) the principal hyperbola through α, β, γ passes through o and, since it also passes through ω_1 and ω_2, we see that triangles $\alpha\beta\gamma$, $o\omega_1\omega_2$ are both inscribed in H. Hence (by Lemma 2) there is a conic H' for which the triangles are both self-polar and a conic H'' to which the triangles are both circumscribed. H' being a conic coaxal with S, and H'' being a parabola touching the axes of S, the necessity part of the theorem is proved.

To prove the sufficiency part, we notice that if triangle $\alpha\beta\gamma$ is self-polar w.r.t. a conic H' for which triangle $o\omega_1\omega_2$ also is self-polar, then (by Lemma 2) there is a conic H passing through α, β, γ, o, ω_1, ω_2. This means that the principal hyperbola through α, β, γ passes through the origin and (by Lemma 1), α, β, γ constitute a pedal triad. Hence the sufficiency of condition (*a*) of Theorem 2 for a pedal triad is established. The sufficiency of condition (*b*) of the theorem is proved similarly.

Vaidyanathaswamy gives [**6**, pp. 301–2] along with the two conditions for a pedal triad in Theorem 1, the two more in the corollary which follows. Condition (*a*) of the corollary is got by reciprocating condition (*a*) of Theorem 1 w.r.t. the

* The abbreviation " w.r.t." is used for " with respect to " here and elsewhere in this article.

conic H' in the proof of the theorem, while condition (b) of the corollary is got by reciprocating condition (b) of Theorem 1 w.r.t. S itself.

COROLLARY 1. *A necessary and sufficient condition for three points on a central conic S to constitute a pedal triad is that either* (a) *the triangle formed by the points should be circumscribed to a coaxal conic, or* (b) *the triangle formed by the tangents at the points should be inscribed in an Apollonian hyperbola.*

3. The geometrical characterisation of pedal triads on a central conic, as in § 2, may be viewed as an interpretation of Burnside's analytical condition for the concurrence of three normals [cf. **5**]. But before we take up this point, it may be well to recall the extension of Burnside's condition to θ-normals and a necessary-sufficient condition for a quasi-normal tetrad on a central conic which is closely related to the extension.

Let us denote the ellipse included in (1) by

$$E \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \qquad \text{(4)}$$

and the point ($a \cos \phi$, $b \sin \phi$) with eccentric angle ϕ on the ellipse by ϕ. Then the equation of the principal hyperbola H meeting E in α, β, γ, δ can be put in the form

$$H \equiv x^2/a^2 + y^2/b^2 - 1 + kuu' = 0, \qquad \text{(2a)}$$

where

$$u \equiv \frac{x}{a} \cos \tfrac{1}{2}(\alpha+\beta) + \frac{y}{b} \sin \tfrac{1}{2}(\alpha+\beta) - \cos \tfrac{1}{2}(\alpha-\beta),$$

$$u' \equiv \frac{x}{a} \cos \tfrac{1}{2}(\gamma+\delta) + \frac{y}{b} \sin \tfrac{1}{2}(\gamma+\delta) - \cos \tfrac{1}{2}(\gamma-\delta),$$

and on comparison of (2) with (2a)

$$\begin{aligned}\{1 + k \cos \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma+\delta)\} : 0 &= \{1 + k \sin \tfrac{1}{2}(\alpha+\beta) \sin \tfrac{1}{2}(\gamma+\delta)\} : 0 \\ &= k \sin \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) : 2ab \\ &= \{-1 + k \cos \tfrac{1}{2}(\alpha-\beta) \cos \tfrac{1}{2}(\gamma-\delta)\} : c.\end{aligned}$$

From the first two of the above ratios we obtain

$$\cos \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma+\delta) = \sin \tfrac{1}{2}(\alpha+\beta) \sin \tfrac{1}{2}(\gamma+\delta),$$

that is,

$$\cos \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) = 0,$$

or

$$\tfrac{1}{2}(\alpha+\beta+\gamma+\delta) = n\pi + \tfrac{1}{2}\pi \quad (n \text{ integral}).$$

And from the last two ratios, in conjunction with the first, we get

$$\begin{aligned}\frac{c}{2ab} \sin \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) &= -\frac{1}{k} + \cos \tfrac{1}{2}(\alpha-\beta) \cos \tfrac{1}{2}(\gamma-\delta) \\ &= \cos \tfrac{1}{2}(\alpha+\beta) \cos \tfrac{1}{2}(\gamma+\delta) + \cos \tfrac{1}{2}(\alpha-\beta) \cos \tfrac{1}{2}(\gamma-\delta) \\ &= \tfrac{1}{2}\{\cos \tfrac{1}{2}(\alpha+\beta+\gamma+\delta) + \cos \tfrac{1}{2}(\alpha+\beta-\gamma-\delta) \\ &\qquad + \cos \tfrac{1}{2}(\alpha+\gamma-\beta-\delta) + \cos \tfrac{1}{2}(\alpha+\delta-\beta-\gamma)\}.\end{aligned}$$

Since $\tfrac{1}{2}(\alpha+\beta+\gamma+\delta) = n\pi + \tfrac{1}{2}\pi$, the last step reduces to

$$(-1)^n c/2ab = \tfrac{1}{2}(-1)^n \{\sin(\alpha+\beta) + \sin(\alpha+\gamma) + \sin(\beta+\gamma)\}.$$

Interpreting c in the above relation geometrically, we have :

THEOREM 2. *If the H in* (2) *is the principal hyperbola through points α, β, γ on*

the E given by (4), *then*

$$ab\ \Sigma \sin(\alpha+\beta) = \begin{cases} \textit{Product of the segments of division, by the origin,} \\ \textit{of the chord of } H \textit{ along } y = x.^* \end{cases}$$

This theorem has long been known [2, p. 539 ; 8, p. 127] in its specialised form involving θ-normals. A. A. Krishnaswami Ayyangar [4] has suggested for it an ingenious proof which, however, is less direct than the one given here. The theorem in its specialised form (appearing below) expresses a necessary and sufficient condition for three points on E to form a quasi-normal triad of deviation θ.

THEOREM 2a. *If α, β, γ are points on E the θ-normals at which concur, then*

$$\Sigma \sin(\beta+\gamma) = \frac{2ab}{b^2-a^2}\cot\theta.$$

Conversely, if θ is determined in terms of α, β, γ by the above relation, then the θ-normals at α, β, γ concur.

Proof. To prove the first part of the theorem we have merely to take for the equation of the hyperbola in Theorem 2 the special form given by (3).

To establish the converse part of the theorem we note that the principal hyperbola H determined by α, β, γ has equation (2) with

$$c = 2a^2b^2 \cot\theta/(b^2-a^2).$$

H can therefore be identified with the Tesch hyperbola in (3) and we can infer at once the concurrence of the θ-normals at α, β, γ.

COROLLARY 2a (Burnside). *The necessary and sufficient condition for α, β, γ to be a pedal triad on E is*

$$\Sigma \sin(\beta+\gamma) = 0.$$

Implicit in the proof of Theorem 2 is another theorem [2, p. 538] which has been neglected of late by textbook writers :

THEOREM 3. *A necessary and sufficient condition for points α, β, γ, δ on E to form a quasi-pedal tetrad (i.e. to be the intersections of E and a principal hyperbola H) is that the points δ and $(\alpha+\beta+\gamma)$ should be images of each other w.r.t. the y-axis.*

Proof. The necessity of the given condition is a consequence of the fact

$$\alpha+\beta+\gamma+\delta = 2n\pi+\pi \quad (n \text{ integral}).$$

The sufficiency of the given condition follows from the fact that, if the principal hyperbola H through α, β, γ meets E again in δ_1, then δ_1 and $(\alpha+\beta+\gamma)$ are images of each other w.r.t. the y-axis, with the result that δ_1 coincides with δ.

The characterisation of a quasi-normal tetrad in Theorem 3 may be supplemented by another, due to Vaidyanathaswamy [7, p. 283], which deserves to be mentioned here as it follows directly from a simple application of tangential equations.

THEOREM 4. *The ϕ-conic of E and H is a parabola which has the equi-conjugate diameters of E for conjugate lines. Conversely, the quasi-pedal tetrad*

* More generally we can prove, in the same way as Theorem 2, a result due to A. A. Krishnaswami Ayyangar [4] : *in Theorem 2, if H is replaced by*

$$d\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) + 2xy + 2gx + 2fy + c = 0,$$

the conclusion of the theorem will be altered to :

$$ab\ \Sigma \sin(\beta+\gamma) + d\ \Sigma \cos(\beta+\gamma) = c.$$

*constituted by the four intersections of E and H may be specified geometrically as the points of contact with E of the common tangents of E and parabolas having the equi-conjugate diameters of E as conjugate lines.**

Joachimsthal's theorem for pedal tetrads has an extension to quasi-pedal tetrads which is an immediate consequence of Theorem 3 and the familiar

LEMMA 3. *A necessary and sufficient condition for points $\alpha, \beta, \gamma, \delta$ on E to be concyclic is that the points δ and $(\alpha+\beta+\gamma)$ should be images of each other w.r.t. the x-axis.*†

The extension of Joachimsthal's theorem which has been referred to is

THEOREM 5. *If the points $\alpha, \beta, \gamma, \delta$ on E constitute a quasi-pedal tetrad (i.e. if the points are the intersections of E and a principal hyperbola H) and δ' is the point on E diametrically opposite to δ, then $\alpha, \beta, \gamma, \delta'$ are concyclic.*

This theorem is given by both Casey [**2**, p. 538] and Baker [**1**, p. 157]. As Hamflett has shown [**3**] it can be reduced to the harmonic property of pole and polar for a circle by means of two successive projections followed by an inversion.

4. The theorems proved in § 3 for the ellipse E can be modified for the hyperbola

$$E' \equiv x^2/a^2 - y^2/b^2 - 1 = 0, \quad\quad (4')$$

by taking any point on E' to be $(a \cosh \phi, b \sinh \phi)$ and referring to it as the point ϕ.‡ The passage from E to E' involves the replacement of b by ib, ϕ by $-i\phi$. Consequently, Theorems 3, 4, 5 as they stand are true for E' while Theorem 2 is true for E' only after the sines in the conclusion are replaced by the corresponding hyperbolic sines. The modification of Theorem 2a for E' is like that of Theorem 2 and expresses the necessary and sufficient condition for the θ-normals at α, β, γ on E' to concur in the form

$$\Sigma \sinh (\beta+\gamma) = \{2ab/(a^2+b^2)\} \cot \theta.$$

5. The theorems discussed above, with the possible exception of Theorem 4, fall within the scope of a first course in analytical conics on traditional lines. If the proof of Theorem 1 in § 2 is considered to be beyond such a course, an analytical proof can be given instead. For instance, in the case in which S is the ellipse E in (4), we can take the equation of a conic coaxal with S as

$$H' \equiv a'x^2 + b'y^2 - 1 = 0 \quad\quad (5)$$

and the equation of a parabola touching the axes of S as

$$H'' \equiv \sqrt{(a''x)} + \sqrt{(b''y)} - 1 = 0. \quad\quad (6)$$

And we can then verify that the necessary and sufficient condition for a triangle $\alpha\beta\gamma$ inscribed in E to be either (a) self-polar w.r.t. H', or (b) circumscribed to H'', assumes (after a little simplification) the form

* Vaidyanathaswamy has also given an interesting *kinematical* characterisation of quasi-pedal tetrads [**7**, § I, pp. 281–2] and, in particular, of pedal tetrads [**6**, § I, p. 299].

† There is a general theorem of Vaidyanathaswamy [**7**, p. 286] which includes both Lemma 3 and Theorem 3.

‡ It is not often realised that, in order to get all the points ϕ on E', we have to suppose that the parameter ϕ is a complex variable restricted to a period strip of $\cosh \phi$, $\sinh \phi$ (*e.g.* $0 \leqslant \text{Im}\,\phi < 2\pi$) and taking there all the values which make $\cosh \phi$, $\sinh \phi$ real. This means, more precisely, that if $\text{Im}\,\phi = 0$ we get points ϕ on one branch of E', while if $\text{Im}\,\phi = \pi$ we get points on the other branch. An obvious consequence of this fact is that a quasi-normal tetrad of points $\alpha, \beta, \gamma, \delta$ on E' cannot all be on the same branch of E', since $-i(\alpha+\beta+\gamma+\delta) = 2n\pi + \pi$ as explained in the present section (§ 4).

$$[\Sigma \sin(\beta+\gamma)]\,[\Sigma \sin(\beta-\gamma)] = 0.^{*} \qquad (7)$$

Since the second factor of the left-hand member of (7) vanishes if and only if two of the three points α, β, γ coincide [**5**, p. 208], our condition (7) is the same as Burnside's in Corollary 2a and the proof of Theorem 1 is complete. This argument can be readily modified, as explained in § 4, to meet the case in which the ellipse E is replaced by the hyperbola E'.

6. It may not be irrelevant to mention in conclusion that the specification of pedal triads in Theorem 1 (a) was first given by Vaidyanathaswamy who also established its distinctness from the specification in Theorem 1 (b) [**6**, § 2, pp. 299–301]. Vaidyanathaswamy's arguments can be presented in such a way that they can be followed easily by anyone who has some acquaintance with three-dimensional geometry. In this presentation we suppose (as we may without loss of generality) that the central conic S under consideration is the ellipse E. Burnside's condition for points α, β, γ on E to be a pedal triad can then be written :

$$\tan\tfrac12\alpha \tan\tfrac12\beta \tan\tfrac12\gamma(\Sigma \tan\tfrac12\beta \tan\tfrac12\gamma) = \Sigma \tan\tfrac12\alpha,$$

or :

$$\left.\begin{array}{c} \Gamma \equiv ZY - X = 0, \\ X = \Sigma \tan\tfrac12\alpha,\ Y = \Sigma \tan\tfrac12\beta \tan\tfrac12\gamma,\ Z = \tan\tfrac12\alpha \tan\tfrac12\beta \tan\tfrac12\gamma. \end{array}\right\} \qquad (8)$$

(8) expresses the fact that pedal triads on E correspond to points on the quadric Γ in the three-dimensional space of (X, Y, Z). This quadric Γ has two systems of generating lines one of which (say, the λ-system) is given by

$$Y = \lambda, \quad Z = X/\lambda, \qquad (9)$$

and the other (the μ-system) by

$$Z = \mu, \quad Y = X/\mu. \qquad (10)$$

Certain simple considerations involving the two systems of generators suffice to bring out the distinction between the triads (a) and (b) of Theorem 1. This distinction is stated in the next theorem whose proof contains incidentally the analytical criteria for the two classes of pedal triads in Theorem 1.

THEOREM 6. (a) *Any λ-generator defined by* (9), *of the quadric Γ in* (8), *corresponds to the family of pedal triads on E specified by the condition* (a) *of Theorem* 1.

(b) *On the other hand, any μ-generator of the quadric Γ, defined by* (10), *corresponds to the family of pedal triads on E specified by the condition* (b) *of Theorem* 1.

Proof. (a) The equations (9) for a λ-generator correspond to

$$\Sigma \tan\tfrac12\beta \tan\tfrac12\gamma = \lambda, \quad \tan\tfrac12\alpha \tan\tfrac12\beta \tan\tfrac12\gamma = \Sigma \tan\tfrac12\alpha/\lambda,$$

or briefly to

$$\Sigma t_\beta t_\gamma = \lambda, \quad t_\alpha t_\beta t_\gamma = \Sigma t_\alpha/\lambda, \quad (t_\alpha = \tan\tfrac12\alpha, \quad t_\beta = \tan\tfrac12\beta, \quad t_\gamma = \tan\tfrac12\gamma),$$

which give, after the elimination of t_α,

$$\lambda(1 + t_\beta^2 t_\gamma^2) - (1+\lambda^2) t_\beta t_\gamma + (t_\beta + t_\gamma)^2 = 0.$$

* The necessary and sufficient condition for a triangle $\alpha\beta\gamma$ inscribed in E to be self-polar w.r.t. H' is actually (11) below ; while the necessary and sufficient condition for triangle $\alpha\beta\gamma$ to be circumscribed to H'' is actually (12). But (11) and (12) can both be reduced to (7).

We can now write down the two relations similar to the last, one of which involves t_γ, t_α, and the other t_α, t_β, and combine all the three relations, obtaining

$$\begin{vmatrix} 1+t_\beta^2 t_\gamma^2 & t_\beta t_\gamma & (t_\beta+t_\gamma)^2 \\ 1+t_\gamma^2 t_\alpha^2 & t_\gamma t_\alpha & (t_\gamma+t_\alpha)^2 \\ 1+t_\alpha^2 t_\beta^2 & t_\alpha t_\beta & (t_\alpha+t_\beta)^2 \end{vmatrix} = 0,$$

or equivalently

$$\begin{vmatrix} (1-t_\beta^2)(1-t_\gamma^2) & t_\beta t_\gamma & (1+t_\beta^2)(1+t_\gamma^2) \\ (1-t_\gamma^2)(1-t_\alpha^2) & t_\gamma t_\alpha & (1+t_\gamma^2)(1+t_\alpha^2) \\ (1-t_\alpha^2)(1-t_\beta^2) & t_\alpha t_\beta & (1+t_\alpha^2)(1+t_\beta^2) \end{vmatrix} = 0.$$

This can be written in the form

$$\begin{vmatrix} \cos\beta\cos\gamma & \sin\beta\sin\gamma & 1 \\ \cos\gamma\cos\alpha & \sin\gamma\sin\alpha & 1 \\ \cos\alpha\cos\beta & \sin\alpha\sin\beta & 1 \end{vmatrix} = 0, \quad \text{.........................(11)}$$

and therefore expresses the fact that the triad $\alpha\beta\gamma$ on E is self-polar w.r.t. a conic H as in (5) coaxal with E. Thus any point on a λ-generator of Γ corresponds to a triad on E specified as in Theorem 1 (a).

(b) The equations (10) for a μ-generator give first

$$t_\alpha t_\beta t_\gamma = \mu, \quad \Sigma t_\beta t_\gamma = \Sigma t_\alpha / \mu$$

and then, after the elimination of t_α,

$$\mu^2(t_\beta+t_\gamma) - \mu(1-t_\beta^2 t_\gamma^2) - t_\beta t_\gamma(t_\beta+t_\gamma) = 0.$$

This relation can be taken in conjunction with the two similar relations involving t_γ, t_α and t_α, t_β, so as to yield

$$\begin{vmatrix} t_\beta+t_\gamma & 1-t_\beta^2 t_\gamma^2 & t_\beta t_\gamma(t_\beta+t_\gamma) \\ t_\gamma+t_\alpha & 1-t_\gamma^2 t_\alpha^2 & t_\gamma t_\alpha(t_\gamma+t_\alpha) \\ t_\alpha+t_\beta & 1-t_\alpha^2 t_\beta^2 & t_\alpha t_\beta(t_\alpha+t_\beta) \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} (t_\beta+t_\gamma)(1+t_\beta t_\gamma) & 1-t_\beta^2 t_\gamma^2 & (t_\beta+t_\gamma)(1-t_\beta t_\gamma) \\ (t_\gamma+t_\alpha)(1+t_\gamma t_\alpha) & 1-t_\gamma^2 t_\alpha^2 & (t_\gamma+t_\alpha)(1-t_\gamma t_\alpha) \\ (t_\alpha+t_\beta)(1+t_\alpha t_\beta) & 1-t_\alpha^2 t_\beta^2 & (t_\alpha+t_\beta)(1-t_\alpha t_\beta) \end{vmatrix} = 0$$

which is the same as

$$\begin{vmatrix} \sec\tfrac{1}{2}(\beta+\gamma) & \operatorname{cosec}\tfrac{1}{2}(\beta+\gamma) & \sec\tfrac{1}{2}(\beta-\gamma) \\ \sec\tfrac{1}{2}(\gamma+\alpha) & \operatorname{cosec}\tfrac{1}{2}(\gamma+\alpha) & \sec\tfrac{1}{2}(\gamma-\alpha) \\ \sec\tfrac{1}{2}(\alpha+\beta) & \operatorname{cosec}\tfrac{1}{2}(\alpha+\beta) & \sec\tfrac{1}{2}(\alpha-\beta) \end{vmatrix} = 0. \quad \text{................(12)}$$

The last is the condition for the triad $\alpha\beta\gamma$ on E to be circumscribed to a parabola H'' like that in (6) touching the axes of E. Hence any point on a μ-generator of Γ corresponds to a triad on E with the specification in Theorem 1 (b).

If, in the proof of Theorem 6 (b), δ is the fourth point on E which, together with α, β, γ, gives rise to a pedal tetrad, we have

$$\tan\tfrac{1}{2}\alpha \tan\tfrac{1}{2}\beta \tan\tfrac{1}{2}\gamma \tan\tfrac{1}{2}\delta = -1,$$

that is,

$$\tan\tfrac{1}{2}\delta = -1/\mu.$$

Since $\tan\tfrac{1}{2}\alpha \tan\tfrac{1}{2}\beta \tan\tfrac{1}{2}\gamma = \mu$, this means that δ is fixed by μ and we have in consequence

Corollary 6 (b). *A μ-generator of the quadric Γ corresponds to a family of pedal triads on E, such that the normals at the points of any triad of the family meet on a fixed normal to E.*

We thus find that the pedal triads on E figuring in Corollary 6 (b) are identical with those in Theorem 1 (b), a fact which can also be established independently without reference to the quadric Γ [**6**, p. 300].

Vaidyanathaswamy has extended to quasi-pedal triads [**8**] his classification of pedal triads on a central conic which we have discussed here. The extension, however, is beyond the scope of the present essay.

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1672. *Potentilla*. . . . Catalogues are filled with strings of potentilla-names, stretching out into infinity like the Pharaohs and each with no more individuality to the uninstructed. . . . I turn up M. Correvon's new work and I read this : " VI ; 20 c : jaune vif ; Eur ; I ; 3." A most searching problem in algebra, evidently, or Rule of Three, or some other high mystery of mathematics.—R. Farrar, *The English Rock-Garden*, Vol. II, p. 88. [Per Mr. W. J. Thompson.]

1673. Jules Partis, a local sculptor, is to try to fly a plane whose propeller is driven by pedals, like a cycle, producing a driving force of from 16 to 20 horse-power.

The plane, designed by Partis, weighs just over 92 lb. and costs about £29. —*News Chronicle*, January 2, 1950. [Per Mr. C. T. Stroud.]

1674. You can not reasonably call a straight line good if the points that compose it are evil.—W. Somerset Maugham, *A Writer's Notebook* (1949), p. 74. [Per Mr. G. E. Crawford.]

1675. " You get to thinking what happens when—you die." " Yes, I suppose you do. Did you—come to any conclusions? " Henry emitted a derisive snort. " I'm a mathematician. They're supposed to be scientists—a sort of bastard scientists. They can't be taken seriously, like chemists and physicists, who put more money in the till. But they sure as hell aren't priests."—Barbara Hunt, *A Little Night Music* [per Mr. G. H. Grattan-Guinness.]

APPLICATION OF MATHIEU'S EQUATION TO STABILITY OF NON-LINEAR OSCILLATOR.

BY N. W. McLACHLAN.

1. *Introduction.*

The differential equation for y, the lateral displacement of an electrically-driven tuning fork,* is

$$\ddot{y} + (c\dot{y}^2 - 2\kappa)\dot{y} + ay = 0, \quad \text{.............................(1)}$$

where $a > 0$, c, κ small > 0. $(c\dot{y}^2 - 2\kappa)$, the coefficient in the damping term, is positive or negative according as $c\dot{y}^2 > 2\kappa$ or $c\dot{y}^2 < 2\kappa$. During periodic motion, the coefficient changes sign, such that the inherent loss per period is compensated exactly by the energy supplied from the driving agent. In the language of " electronics ", loss corresponds to a " positive " resistance and energy supply to a " negative " resistance, both of which vary periodically. Writing (1) in the form

$$\ddot{y} + (c\dot{y}^2)\dot{y} + ay = 2\kappa\dot{y}, \quad \text{.................................(2)}$$

gives the equation for a dissipative system with variable damping coefficient $c\dot{y}^2$, driven by an external force $2\kappa\dot{y}$. When the latter is removed, $\kappa = 0$, and we get

$$\ddot{y} + (c\dot{y}^2)\dot{y} + ay = 0, \quad \text{.................................(3)}$$

which is the equation for the " free " damped vibrations of the tuning fork. If y_0 is the initial amplitude and θ_0 an arbitrary initial phase angle, the first approximation solution of (3) is†

$$y = y_0 \sin(\omega_0 t + \theta_0)/(1 + 3c\omega_0^2 y_0^2 t/4), \quad \text{.......................(4)}$$

where $\omega_0^2 = a$. Herein we shall investigate the stability of motion of the driven fork.

2. *Periodic solution of* (1), § 1.

If we put $y = (2\kappa/c)^{\frac{1}{2}} x$, $2\kappa = \epsilon$ in (1), § 1, it becomes

$$\ddot{x} + \epsilon(\dot{x}^2 - 1)\dot{x} + ax = 0. \quad \text{.............................(1)}$$

We shall use the perturbation method, as in N.D.E., and assume that with ϵ small

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots, \quad \text{.............................(2)}$$

$$a = \omega_0^2 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots, \quad \text{.............................(3)}$$

and take initial conditions $x(0) = A$, $\dot{x}(0) = 0$. Then with $\psi = \omega_0 t$, to order two in ϵ, we find that

$$x = (2/\omega_0\sqrt{3})\{(1 - 5\epsilon^2/192\omega_0^2)\cos\psi - (\epsilon/24\omega_0)(3\sin\psi - \sin 3\psi) + (\epsilon^2/192\omega_0^2)(6\cos 3\psi - \cos 5\psi)\}, \quad \text{.........................(4)}$$

$$y = (2/\omega_0)(2\kappa/3c)^{\frac{1}{2}}\{(1 - 5\kappa^2/48\omega_0^2)\cos\psi - (\kappa/12\omega_0)(3\sin\psi - \sin 3\psi) + (\kappa^2/48\omega_0^2)(6\cos 3\psi - \cos 5\psi)\}, \quad \text{.............................(5)}$$

and

$$a = \omega_0^2 + \kappa^2/2, \quad \text{or} \quad \omega_0 \simeq a^{\frac{1}{2}}(1 - \kappa^2/4a). \quad \text{.....................(6)}$$

3. *The variational equation.*

Let x in (1), § 2, be increased or decreased by a very small amount v, such

* Rayleigh, *Phil. Mag.*, 15, 229, 1883.

† McLachlan, *Ordinary Non-linear Differential Equations* (Oxford, 1950), which will be designated by N.D.E.

that terms in $\dot{v}^2$, $\dot{v}^3$ are negligible. Substituting $(x+v)$ for x yields

$$[\ddot{x}+\epsilon(\dot{x}^2-1)\dot{x}+ax]+\{\epsilon(3\dot{x}+\dot{v})\dot{v}^2\}+\ddot{v}+\epsilon(3\dot{x}^2-1)\dot{v}+av=0. \quad\text{.........(1)}$$

Then [ ] = 0 by (1), § 2, { } is negligible by hypothesis, and there remains the variational equation

$$\ddot{v}+\epsilon(3\dot{x}^2-1)\dot{v}+av=0, \quad\text{.................................(2)}$$

which is linear in v. If the solution of (2) is bounded (unbounded), that of (1), § 1, is stable (unstable), for the procedure is equivalent to determining whether or not the tuning fork moves towards or away from its original configuration after being disturbed. The next step is to derive a Mathieu equation, since the stability properties of its solutions are known.

In a first approximation we may take the solution of (1), § 2, to be

$$x=A\cos\omega_0 t,$$

and we shall determine A for stable operation. Substituting into (2) gives

$$\ddot{v}+\epsilon\{(3\omega_0{}^2A^2/2)-1-(3\omega_0{}^2A^2/2)\cos 2\omega_0 t\}\dot{v}+\omega_0{}^2v=0. \quad\text{............(3)}$$

4. *Solution of* (3), § 3.

With $\omega_0 t=z$, this becomes

$$v''+(\alpha-2\beta\cos 2z)v'+v=0, \quad\text{..............................(1)}$$

where $\alpha=(\epsilon/\omega_0)(3\omega_0{}^2A^2/2-1)$, and $2\beta=3\epsilon\omega_0A^2/2$. Putting

$$v=u(z)\times\exp\{-(\alpha z-\beta\sin 2z)/2\}$$

to remove the middle term, and discarding terms in ϵ^2, leads to the equation

$$u''+(1-2\beta\sin 2z)u=0. \quad\text{...............................(2)}$$

If we write $(x+\pi/4)$ for z and q for β, (2) becomes

$$u''+(1-2q\cos 2x)u=0, \quad\text{...............................(3)}$$

which is the standard form of Mathieu's equation.*

By M.F., Fig. 11, p. 98, the parametric point $(1, q)$ lies in an unstable region of the (a, q) plane, so by (3), 2°, § 4.70, that solution of (3) which is unstable in $x>0$, is

$$u_1(x)=e^{\mu x}\times\text{bounded function having period }2\pi, \quad\text{..............(4)}$$

where $\mu>0$. Hence the solution of (1), appropriate to our investigation, takes the form

$$v=C\exp\{\phi(z)\}\times\text{bounded periodic function}, \quad\text{.................(5)}$$

with $C=e^{-\mu\pi/4}$, and

$$\phi(z)=(\mu-\alpha/2)z+(\beta/2)\sin 2z. \quad\text{...........................(6)}$$

5. *Stability criteria.*

To investigate stability we consider the non-periodic part of $\phi(z)$, namely,

$$\chi(z)=(\mu-\alpha/2)z=\mu+(\epsilon/2\omega_0)(1-3\omega_0{}^2A^2/2). \quad\text{..................(1)}$$

Now $q=\beta=3\epsilon\omega_0A^2/4$, and by M.F. (3), § 4.92, since q is small,

$$\mu=q/2-O(q^3). \quad\text{..(2)}$$

Substituting into (1) yields

$$\chi=(\epsilon/2\omega_0)\{(1-3\omega_0{}^2A^2/4)-O(\epsilon^2)\}. \quad\text{.........................(3)}$$

* McLachlan, *Mathieu Functions* (Oxford, 1947), designated by M.F.

For (i) stability $\chi < 0$, (ii) instability $\chi > 0$, and (iii) neutrality, *i.e.* periodic motion, $\chi = 0$. Hence the requisite conditions for (1), § 2, and (1), § 1, are, respectively,

$$\text{(i)}\ A^2 > 4/3\omega_0^2 - 0(\epsilon^2): \quad \bar{A}^2 > 8\kappa/3\omega_0^2 c - O(\epsilon^2); \qquad (4)$$

$$\text{(ii)}\ A^2 < 4/3\omega_0^2 - 0(\epsilon^2); \quad \bar{A}^2 < 8\kappa/3\omega_0^2 c - O(\epsilon^2); \qquad (5)$$

$$\text{(iii)}\ A^2 = 4/3\omega_0^2 - 0(\epsilon^2); \quad \bar{A}^2 = 8\kappa/3\omega_0^2 c - O(\epsilon^2), \qquad (6)$$

with $\omega_0^2 \simeq a$. The stable amplitude is given by (6), which agrees with the coefficients of $\cos\psi$ in (4), (5), § 2. If $\bar{A}$ either exceeds or is less than the value given by (6), then from (4), (5) the amplitude tends to its value during periodic motion, so the system is stable.

In a second approximation using additional terms in (4), (5), § 2, and retaining terms in ϵ^2, etc., the equation corresponding to (3), § 4, will be a Hill type (M.F., Chap. VI). The stability can be investigated in the above way, using the appropriate expression for $\chi(z)$.

6. *The energy equation.*

Write $v = dy/dt$, $d^2y/dt^2 = v\,dy/dy$, and (2), § 1, becomes

$$v\,dv/dy + ay + cv^3 = 2\kappa v. \qquad (1)$$

Multiplying throughout by dy and integrating over a period $t = (0, \tau)$, with $\tau = 2\pi/\omega_0$, we obtain the energy equation

$$\Big[v^2/2 + ay^2/2\Big]_0^{t=\tau} + c\int_0^\tau v^4\,dt = 2\kappa\int_0^\tau v^2\,dt. \qquad (2)$$

The first and second terms pertain to kinetic and potential energy, respectively, and in virtue of periodicity, both of them vanish on inserting the limits. The first integral represents the inherent energy loss, while the second represents an equal amount of energy supplied by the driving agent by way of compensation. Using the first approximation $y = \bar{A}\cos\omega_0 t$, $v = -\omega_0\bar{A}\sin\omega_0 t$, leads to the result

$$3\pi c\omega_0^3\bar{A}^4/4 = 2\kappa\pi\omega_0\bar{A}^2, \qquad (3)$$

or energy loss per period = energy supplied p.p. ....(4

From (3) we obtain

$$\bar{A} = (2/\omega_0)(2\kappa/3c)^{\frac{1}{2}}, \qquad (5)$$

which is the same as at (5), § 2, to the order of approximation contemplated.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

MATHEMATICAL NOTES

2201. *Pandiagonal and symmetrical magic squares.*

Rouse Ball, in Chapter VII of his *Mathematical Recreations and Essays* (I quote from the eleventh edition, of 1939), discusses pandiagonal and symmetrical magic squares, but does not consider at all fully in what cases a magic square can possess both these additional properties simultaneously. It has been assumed in two recent papers in the *Gazette* that a magic square of even order can have the pandiagonal property only trivially, in the sense defined below. If so, the two properties could not be combined, at least not with the numbers all unequal, for squares of even order. This note will show incidentally that the result referred to is incorrect if the order is divisible by 8 ; it is also incorrect if the order is divisible by 4, though I think it is true if the order is oddly even. For brevity I confine myself to the easiest case, where the order is $8n$, n integral.

2. I denote the cells of the square by integral coordinates x, y, supposed reduced (mod $8n$). The rows, columns, and generalised diagonals along which the sum has to be magic are then given by putting x, y or $x \pm y$ congruent to a constant (mod $8n$). Denoting the magic sum by S, the row and column properties are expressed by

$$\sum_{x=0}^{8n-1} a(x, y) = S \quad \text{(for all } y\text{)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\sum_{y=0}^{8n-1} a(x, y) = S \quad \text{(for all } x\text{)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$a(x, y)$ denoting the number in the cell (x, y). The pandiagonal properties are

$$\sum_{x=0}^{8n-1} a(x, \quad x+i) = S \quad \text{(for all } i\text{)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\sum_{x=0}^{8n-1} a(x, \ -x+i) = S \quad \text{(for all } i\text{)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

3. Now the cells (x, y) and $(x+4n, y+4n)$ come on the same diagonal in either system, for we have

$$x+4n \pm (y+4n) \equiv x \pm y \pmod{8n}.$$

Hence the square is obviously, and we may say trivially, pandiagonal if for all x, y we have

$$a(x, y) + a(x+4n, y+4n) = S/4n. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The cell symmetrically related to $z(x, y)$ is $(8n-1-x, 8n-1-y)$, and the square is therefore symmetrical if

$$a(x, y) + a(8n-1-x, 8n-1-y) = S/4n. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

(5) and (6) together would imply the equality of numbers in different cells, so on showing that the equations (1) to (4) and (6) can be satisfied with the a equal to the distinct integers 0 to $64n^2-1$, which is the main object of this note, I shall have shown incidentally that a pandiagonal magic square of order $8n$ does not necessarily satisfy (5), as assumed in the papers above referred to (in order to prove that the determinant of the square is zero).

4. My method is an extension of that of Margossian, described at pp. 208–10 of Rouse Ball. I first construct a preliminary square with elements $A(x, y)$ defined by

$$A\,(x,\,y)=8nr\,(x,\,y)+s\,(x,\,y), \qquad\qquad (7)$$

$$r\,(x,\,y)\equiv \quad x+2ny+n, \qquad\qquad (8)$$

$$s\,(x,\,y)\equiv 2nx+ \quad y+n, \quad (\text{mod } 8n), \qquad\qquad (9)$$

$$0\leqslant r,\, s\leqslant 8n-1. \qquad\qquad (10)$$

Since the determinant of the coefficients of x, y in (8) and (9) is prime to $8n$, the pairs of values of r, s are all different and the A are the numbers 0 to $64n^2-1$. On moving from one cell to the next along any row, column, or generalised diagonal, either r or s increases by ± 1, $\pm 2n$, or $\pm 2n\pm 1$. Hence along the whole row, column or diagonal, r or s either runs through all values from 0 to $8n-1$, or takes just four values differing by multiples of $2n$, each repeated $2n$ times. The square thus lacks some of the required properties; but it possesses that of symmetry, since (8) gives

$$r\,(x,\,y)+r\,(8n-1-x,\,8n-1-y)$$
$$\equiv 8n-1+2n\,(8n-1)+2n\equiv -1 \;\; (\text{mod } 8n),$$

whence by (10)

$$r\,(x,\,y)+r\,(8n-1-x,\,8n-1-y)=8n-1.$$

s satisfies a similar relation, hence by (7)

$$A\,(x,\,y)+A\,(8n-1-x,\,8n-1-y)=64n^2-1$$

and evidently $S=4n\,(64n^2-1)$.

5. I now put

$$u=u\,(x,\,y)=8n-1-r\,(x,\,y), \qquad\qquad (11)$$
$$\text{if } \; 2n\leqslant r\leqslant 6n-1,$$
$$u=r, \text{ otherwise.}$$

I define $v=v\,(x,\,y)$ similarly in terms of s, and put

$$a\,(x,\,y)=8nu\,(x,\,y)+v\,(x,\,y). \qquad\qquad (12)$$

Now I show that the square formed with the numbers $a\,(x,\,y)$ has all the desired properties.

6. Firstly, the a are the unequal numbers 0 to $64n^2-1$. For, the pairs r, s being all different, it is readily seen that the pairs u, v are all different. ((11) transforms unequal r into unequal u.) Next, (11) transforms two r's whose sum is $8n-1$ into two u's whose sum is $8n-1$, and similarly for s, v. Hence the a square retains the symmetry property of the A square. It remains to demonstrate that (1) to (4) hold, and it suffices to show that they hold for u and v separately (with a different magic sum $4n\,(8n-1)$). I need only consider u. This is obvious in a sum in which r, and hence also u, runs through all the values 0 to $8n-1$ once each. When r takes four values R, $R+2n$, $R+4n$, $R+6n$, u by (11) takes the values

$$R,\; 6n-1-R,\; 4n-1-R,\; R+6n,$$

whose sum is $16n-2$, $2n$ times each, giving the correct magic sum $4n\,(8n-1)$. This completes the proof.

G. L. WATSON.

2202. *Evaluation of complex roots of an algebraic equation.*

$$\text{Let } f(x)\equiv a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0 \qquad\qquad (1)$$
$$=g(x)(x^2+px+q)+b_{n-1}x+(b_n+pb_{n-1}) \qquad\qquad (2)$$

where

$$g(x)=b_0x^{n-2}+b_1x^{n-3}+\ldots+b_{n-3}x+b_{n-2} \qquad\qquad (3)$$
$$=h(x)(x^2+px+q)+c_{n-3}x+(c_{n-2}+pc_{n-3}) \qquad\qquad (4)$$

where

$$h(x) = c_0x^{n-4} + c_1x^{n-5} + \ldots + c_{n-5}x + c_{n-4}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

We may regard b_{n-1} and b_n as functions of p and q. The correct values of p and q make b_n and b_{n-1} zero. Let these correct values be p_0 and q_0, so that

$$x^2 + p_0x + q_0 = 0.$$

Now by Taylor's theorem,

$$b_{n-r} \simeq \left(\frac{\partial b_{n-1}}{\partial p}\right)_0 (-\delta p) + \left(\frac{\partial b_{n-1}}{\partial q}\right)_0 (-\delta q), \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and

$$b_n \simeq \left(\frac{\partial b_n}{\partial p}\right)_0 (-\delta p) + \left(\frac{\partial b_n}{\partial q}\right)_0 (-\delta q) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where

$$b_{n-1} \equiv b_{n-1}(p_0 - \delta p,\ q_0 - \delta q),\ (b_{n-1})_0 = b_{n-1}(p_0, q_0) = 0,$$
$$b_n \equiv b_n(p_0 - \delta p,\ q_0 - \delta q), \qquad (b_n)_0 = b_n(p_0,\ q_0) \quad = 0.$$

From (2) and (4)

$$(x+p)b_{n-1} + b_n = -h(x)(x^2+px+q)^2 - (c_{n-3}x + c_{n-2} + pc_{n-3})(x^2+px+q) \quad \ldots(8)$$

Differentiate (8) partially with respect to p:

$$(x+p)\frac{\partial b_{n-1}}{\partial p} + b_{n-1} + \frac{\partial b_n}{\partial p} = -h(x)(x^2+px+q) \,.\, 2x - (x^2+px+q)^2 \frac{\partial h(x)}{\partial p}$$
$$- c_{n-3}(x^2+px+q) - x(c_{n-3}x + c_{n-2} + pc_{n-3})$$

and since $(b_{n-1})_0 = 0$,

$$(x+p_0)\left(\frac{\partial b_{n-1}}{\partial p}\right)_0 + \left(\frac{\partial b_n}{\partial p_0}\right) = -c_{n-3}x^2 - xc_{n-2} - xp_0c_{n-3},$$

also

$$x^2 = -p_0x - q_0.$$

Thus

$$(x+p_0)\left(\frac{\partial b_{n-1}}{\partial p}\right)_0 + \left(\frac{\partial b_n}{\partial p}\right)_0 = -x\,(c_{n-2} + p_0c_{n-3}) + c_{n-3}p_0x + c_{n-3}q_0$$
$$= -c_{n-2}x + c_{n-3}q_0.$$

Hence

$$\left(\frac{\partial b_{n-1}}{\partial p}\right)_0 = -c_{n-2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

and

$$p_0\left(\frac{\partial b_{n-1}}{\partial p}\right)_0 + \left(\frac{\partial b_n}{\partial p}\right)_0 = c_{n-3}q_0,$$

or

$$\left(\frac{\partial b_n}{\partial p}\right)_0 = c_{n-3}q_0 + c_{n-2}p_0$$
$$\simeq c_{n-3}q + c_{n-2}p = -c_{n-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

for convenience in tabulation.

Differentiating (8) partially with respect to q, similar calculations lead to

$$\left(\frac{\partial b_{n-1}}{\partial q}\right)_0 = -c_{n-3}, \quad \left(\frac{\partial b_n}{\partial q}\right)_0 = -c_{n-2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Substituting from (9), (10) and (11) in (6) and (7), we have

$$b_{n-1} + c_{n-2}(-\delta p) + c_{n-3}(-\delta q) = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6a)$$

$$b_n + c_{n-1}(-\delta p) + c_{n-2}(-\delta q) \quad = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

From ($6a$) and ($7a$) the corrections to the assumed values of p and q may be calculated. It is a simple matter to perform synthetic division of a polynomial by the quadratic $x^2 + px + q$, using a calculating machine, and the above method, which is outlined by W. E. Milne in his *Numerical Calculus* (Princeton University Press, 1949), provides a useful alternative to Graeffe's method. Milne suggests that complex roots may be evaluated by Newton's method, but this would be more cumbersome than the method he gives as above.

The use of the quadratic factor in the solution of algebraic equations offers considerable possibilities, which seem to have been overlooked in most textbooks.

M. Bridger.

2203. *Differentials.*

1. I have read the articles on this topic by A. Barton (Vol. XXIX, No. 287, p. 193), E. G. Phillips (XXXIII, No. 305, p. 202) and J. Hadamard (XXXIV, No. 309, p. 210) with much interest. I take the view that differentials are not increments. It will assist me in putting forward my own views if I first outline a development of the theory. It will be convenient and sufficient for my purpose to consider functions of one and two variables only and to assume continuity and differentiability as and when required. Following the usual convention, all numbers ϵ tend to zero with the increments.

2. If $z = f(x)$,

$$\frac{dz}{dx} = f'(x), \quad \frac{\Delta z}{\Delta x} = f'(x) + \epsilon,$$

and $\quad \Delta z = \{f'(x) + \epsilon\}.\Delta x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (i)

2.1. If $\quad z = f(x,y)$,

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x,y)\\ &= \{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)\} + \{f(x, y + \Delta y) - f(x,y)\}\\ &= \{f_x(x, y + \Delta y) + \epsilon_1\}.\Delta x + \{f_y(x,y) + \epsilon_2\}.\Delta y, \text{ on applying (i) or the first mean value theorem twice,}\\ &= \{f_x(x,y) + \epsilon_3\}.\Delta x + \{f_y(x,y) + \epsilon_2\}.\Delta y \text{ by continuity of } f_x.\end{aligned}$$

Thus $\quad \Delta z = (f_x + \epsilon_3).\Delta x + (f_y + \epsilon_2).\Delta y. \quad \ldots\ldots\ldots\ldots\ldots\ldots$ (ii)

It is important to note that equation (ii) is valid whether or not x and y are independent variables.

2.2. Defining differentiability of a function of one variable by the equation

$$\Delta z = (A + \epsilon).\Delta x,$$

and of a function of two variables by the equation

$$\Delta z = (A + \epsilon_1).\Delta x + (B + \epsilon_2).\Delta y,$$

it follows at once that A and B are derivatives and that for a function of one variable possession of a derivative and differentiability are equivalent.

2.3. The situations expressed by equations (i) and (ii) are *summarised* by writing

$$dz = f'(x).dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

and $\quad dz = f_x.dx + f_y.dy. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (iv)

2.4. From equation (ii), if x, y are functions of variables t,u then

$$\frac{\Delta z}{\Delta t} = (f_x + \epsilon_3)\frac{\Delta x}{\Delta t} + (f_y + \epsilon_2)\frac{\Delta y}{\Delta t};$$

whence

$$\frac{\partial z}{\partial t} = f_x \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t} \cdot \qquad \text{(v)}$$

Similarly, if x,y are functions of t only, we have merely to replace ∂ by d in (v). In the special case $t = x$, so that z is a function of variables x,y where y is a function of x, we have

$$dx/dt = dx/dx = 1,$$

and

$$\frac{dz}{dx} = f_x + f_y \cdot \frac{dy}{dx}.$$

2.5. Note that all the results of 2.4 follow at once on dividing equation (iv) by dt (or dx), partial derivative notation being used where the context demands it.

3. The fundamental point of the differential notation is that it provides an *algebraical* device for *summarising* an essentially *analytical* (limiting) process. It is a symbolism, manipulated according to certain rules, which enables one to avoid the details of the limiting process. This is seen in 2.5.

The economy of the differential notation is two-fold, in that the same differential form provides a *prescription* for derivatives with respect to several parameters. Thus, if equation (iv) is divided by du, instead of dt, $\partial z/\partial u$ is at once obtained. The *invariance with respect to parameter* of the differential notation is then the second reason for its power and usefulness. A differential form, such as equation (iv), is an incomplete algebraical symbolism, prescribing a derivative with respect to an arbitrary parameter.

3.1. As illustrations of the points of the preceding paragraph, consider the equations

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= dr^2 + r^2 \,.\, d\theta^2, \quad \text{for arc length;} \\ dS &= 2\pi y \,.\, ds, \quad \text{for surface area;} \\ d(uv) &= u \,.\, dv + v \,.\, du, \\ \int u \,.\, dv &= uv - \int v \,.\, du, \quad \text{for integration by parts.} \end{aligned}$$

In the last example an integral with respect to an arbitrary parameter is prescribed.

3.2. The rules of operation of differentials are governed entirely by the limiting processes which they summarise and not in the least by any theory of what differentials *really are*. In the same way, the arithmetic of the integers is governed by the Peano axioms and not by any theory of what integers *really are*. Thus it is unnecessary to define what the differential dx, in isolation, means. The view that it is an increment leads to difficulties to which I see no satisfactory answer. Two of these are indicated in paragraphs 4.4 and 4.5; another concerns the difference in status between the differentials of dependent and independent variables, which is aesthetically most unsatisfying.

3.3. For purposes of numerical approximation it is clear, from equations (i) and (ii), that equations (iii) and (iv) yield correct results if increments are *substituted* for dx and dy. The legitimacy of this substitution does not, however, imply that differentials *are* increments. In modern algebra one

considers polynomials in an *indeterminate* x, with coefficients in a field or ring. It is not assumed that x is a *variable* member of the field. Nevertheless, the properties of polynomials so obtained lead to valid properties of the corresponding polynomial functions if a variable member x of the field is substituted for the indeterminate x. (See, for example, Birkhoff and MacLane, *A Survey of Modern Algebra*, Ch. IV ; or MacDuffee, *Vectors and Matrices*, Ch. IV.)

4. We may obtain $\partial^2 z/\partial t^2$ from equation (v) by differentiating the two products partially with respect to t. In this process f_x and f_y are treated in the same manner as was f originally ; that is, the x and y derivatives are taken and multiplied by $\partial x/\partial t$ and $\partial y/\partial t$ respectively. Thus

$$\frac{\partial^2 z}{\partial t^2}=f_x\cdot\frac{\partial^2 x}{\partial t^2}+\frac{\partial x}{\partial t}\left\{f_{xx}\cdot\frac{\partial x}{\partial t}+f_{xy}\cdot\frac{\partial y}{\partial t}\right\}$$

$$+f_y\cdot\frac{\partial^2 y}{\partial t^2}+\frac{\partial y}{\partial t}\left\{f_{yx}\cdot\frac{\partial x}{\partial t}+f_{yy}\cdot\frac{\partial y}{\partial t}\right\}$$

$$=f_x\cdot\frac{\partial^2 x}{\partial t^2}+f_y\cdot\frac{\partial^2 y}{\partial t^2}+f_{xx}\left(\frac{\partial x}{\partial t}\right)^2+2f_{xy}\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right)+f_{yy}\left(\frac{\partial y}{\partial t}\right)^2, \quad \text{.....(vi)}$$

taking $f_{xy}=f_{yx}$.

4.1. If $z=uv$, where u and v are functions of *one or more* variables $x,y, \dots$, then

$$\Delta z=(u+\Delta u)(v+\Delta v)-uv$$
$$=u\,.\,\Delta v+v\,.\,\Delta u+\Delta u\,.\,\Delta v,$$

whence

$$\frac{dz}{dx}=u\,.\,\frac{dv}{dx}+v\,.\,\frac{du}{dx}\,.$$

This is summarised in the differential notation

$$dz=u\,.\,dv+v\,.\,du. \quad \text{..............................(vii)}$$

4.2. On applying (vii) *tentatively* to (iv) and writing $d(dz)=d^2z$, and so on, we obtain

$$d^2z=f_x\,.\,d^2x+dx\,.\,d(f_x)+f_y\,.\,d^2y+dy\,.\,d(f_y)$$
$$=f_x\,.\,d^2x+f_y\,.\,d^2y+dx\{f_{xx}\,.\,dx+f_{xy}\,.\,dy\}+dy\{f_{yx}\,.\,dx+f_{yy}\,.\,dy\}$$
$$=f_x\,.\,d^2x+f_y\,.\,d^2y+f_{xx}\,.\,dx^2+2f_{xy}\,.\,dx\,.\,dy+f_{yy}\,.\,dy^2 \quad \text{...............(viii)}$$

taking $f_{xy}=f_{yx}$.

4.3. On dividing (viii) by dt^2 and changing to partial notation we again obtain (vi). This indicates that the *tentative formal operation* of 4.2 is a useful one leading to valid results. For this, and for no other reason, it is incorporated in the theory of differentials.

4.4. In the special case $t=x$, when z is a function of variables x,y where y is a function of x (and of u, possibly) the term d^2x/dx^2 (or $\partial^2x/\partial x^2$) in equation (vi) is zero. Hence the corresponding term in equation (viii) may be omitted and we obtain the rule of procedure that if x is an independent variable, $d^2x=0$. I am not satisfied by the explanation of this rule offered by the " increment theory " of differentials, that dx is a constant in this case.

It is to be noted that d^2x is also zero if x is a linear function, $x=at+bu$, of independent variables t,u.

4.5. If $z=f(x)$, where x is a function of t, then

$$dz=f'(x)\,.\,dx$$

and

$$d^2z=f''(x)\,.\,dx^2+f'(x)\,.\,d^2x,$$

whence, on division by dt^2

$$\frac{d^2z}{dt^2}=f''(x)\cdot\left(\frac{dx}{dt}\right)^2+f'(x)\cdot\frac{d^2x}{dt^2}$$

$$=\frac{d^2z}{dx^2}\cdot\left(\frac{dx}{dt}\right)^2+\frac{dz}{dx}\cdot\frac{d^2x}{dt^2}\cdot \quad\text{.........................(ix)}$$

It is *not* true that

$$\frac{d^2z}{dt^2}=\frac{d^2z}{dx^2}\left(\frac{dx}{dt}\right)^2. \quad\text{.................................(x)}$$

On the present theory the invalidity of (x) is easily explained ; for the rules of operation lead to (ix) and not to (x). But the " increment theory " offers difficulties. For, if the differentials of independent variables are increments, they and the differentials of dependent variables (which are defined in terms of them by equations such as (iii) and (iv)) obey the laws of arithmetic. Hence it would appear that the quantities dx^2 on the right-hand side of (x) cancel and that (x) is the valid and (ix) the invalid equation. P. GANT.

2204. *A proof that there is no triangle the magnitudes of whose sides, area and medians are integers.*

It has been conjectured by J. Travers, that there is no triangle whose sides, medians, and area are positive integers. (Note 2084.) This conjecture is shown to be true in the present note.

The existence of a triangle of the above type implies, and is implied by, the existence of a triangle whose sides, medians and area are rational numbers. Suppose that there were such a triangle say ABC. Let the sides opposite A, B, C, be of lengths a, b, c, and the medians through A, B, C, be of lengths l, m, n, respectively. Let the length of the perpendicular from A to BC be h and the distance of the foot of this perpendicular from B be x. It is assumed that the letters A, B, C, have been allocated so that the foot of this perpendicular lies between B and C and that $x\leqslant\frac{1}{2}a$.

By Pythagoras,

$$c^2=h^2+x^2.$$

$$l^2=h^2+(\tfrac{1}{2}a-x)^2.$$

Subtracting, $l^2-c^2=\frac{1}{4}a^2-ax$. Thus x is rational. h is rational because the area of ABC and a are rational.

Hence if a required type triangle exists, then the following equations have solutions a, b, c, l, m, n, h, x, which are rational numbers.

$$\left.\begin{aligned} l^2&=h^2+(\tfrac{1}{2}a-x)^2\\ c^2&=h^2+x^2.\\ b^2&=h^2+(a-x)^2\\ 4m^2&=h^2+(a+x)^2\\ 4n^2&=h^2+(2a-x)^2\end{aligned}\right\}\quad\text{..............................(1)}$$

By multiplying by an appropriate integer it follows that these equations have solutions which are integers. We shall show that this is impossible.

The only positive integral solutions of the equation $\alpha^2+\beta^2=\gamma^2$ are of the form $\alpha=\lambda\mu$, $\beta=\frac{1}{2}(\lambda^2-1)\mu$, $\gamma=\frac{1}{2}(\lambda^2+1)\mu$ where λ and μ are positive integers and either μ is even or λ is odd (or both). From equations (1),

$$h=\lambda_1\mu_1=\lambda_2\mu_2=\lambda_3\mu_3=\lambda_4\mu_4=\lambda_5\mu_5,$$

$$\tfrac{1}{2}a-x=\tfrac{1}{2}(\lambda_1^2-1)\mu_1,$$

$$x=\tfrac{1}{2}(\lambda_2^2-1)\mu_2,$$

$$a - x = \tfrac{1}{2}(\lambda_3^2 - 1)\mu_3,$$
$$a + x = \tfrac{1}{2}(\lambda_4^2 - 1)\mu_4,$$
$$2a - x = \tfrac{1}{2}(\lambda_5^2 - 1)\mu_5.$$

Hence

$$2a = \tfrac{1}{2}(\lambda_2^2 - 1)\mu_2 + \tfrac{1}{2}(\lambda_5^2 - 1)\mu_5,$$
$$2a = \tfrac{1}{2}(\lambda_3^2 - 1)\mu_3 + \tfrac{1}{2}(\lambda_4^2 - 1)\mu_4,$$
$$\tfrac{1}{2}a = \tfrac{1}{2}(\lambda_1^2 - 1)\mu_1 + \tfrac{1}{2}(\lambda_2^2 - 1)\mu_2.$$

Express μ_i in terms of h and λ_i. Then

$$(\lambda_2 - \lambda_2^{-1}) + (\lambda_5 - \lambda_5^{-1}) = (\lambda_3 - \lambda_3^{-1}) + (\lambda_4 - \lambda_4^{-1}), \qquad \text{(2)}$$
$$(\lambda_2 - \lambda_2^{-1}) + (\lambda_5 - \lambda_5^{-1}) = 4\{(\lambda_1 - \lambda_1^{-1}) + (\lambda_2 - \lambda_2^{-1})\}. \qquad \text{(3)}$$

Consider (2). λ_i is a positive integer, thus the integral part of the left-hand side is

$$\left.\begin{array}{rll} \lambda_2 + \lambda_5 - 1 & \text{if} & \lambda_2 \geqslant 2,\ \lambda_5 \geqslant 2 \\ \lambda_5 - 1 & \text{if} & \lambda_2 = 1,\ \lambda_5 \geqslant 2 \\ 0 & \text{if} & \lambda_2 = \lambda_5 = 1 \end{array}\right\} \qquad \text{(4)}$$

If $\lambda_i \geqslant 2, i = 2, 3, 4, 5$, then (4) gives

$$\lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 \qquad \text{(5)}$$

and from (2)

$$\lambda_2^{-1} + \lambda_5^{-1} = \lambda_3^{-1} + \lambda_4^{-1}. \qquad \text{(6)}$$

Thus $\lambda_2\lambda_5 = \lambda_3\lambda_4$. Hence λ_2, λ_5 and λ_3, λ_4 are roots of the same quadratic and $\lambda_2 = \lambda_3, \lambda_5 = \lambda_4$ (or vice-versa according to the notation).

If $\lambda_2 = 1, \lambda_i \geqslant 2, i = 3, 4, 5$, (4) gives $\lambda_5 = \lambda_3 + \lambda_4$ and from (2) $\lambda_5^{-1} = \lambda_3^{-1} + \lambda_4^{-1}$. Thus $\lambda_5^2 = \lambda_3\lambda_4 = (\lambda_3 + \lambda_4)^2$ and $\lambda_3^2 + \lambda_4^2 + \lambda_3\lambda_4 = 0$. This is impossible.

If $\lambda_2 = 1$ and $\lambda_5 = 1$ then by (2) $\lambda_3 = \lambda_4 = 1$.

If $\lambda_2 = 1, \lambda_3 = 1$, then by (4) $\lambda_5 = \lambda_4$.

Hence in all cases,

$$\lambda_2 = \lambda_3,\ \lambda_4 = \lambda_5.$$

(The case $\lambda_2 = \lambda_4$, $\lambda_3 = \lambda_5$ cannot arise as it implies $a = 0$.)

From the expressions for a and x, $\lambda_2 = \lambda_3$ implies that $x = \tfrac{1}{2}a$. Hence $\lambda_1 = 1 (\mu \neq 0$ as $h \neq 0)$. (3) now gives

$$\lambda_5 - \lambda_5^{-1} = 3(\lambda_2 - \lambda_2^{-1}). \qquad \text{(7)}$$

If $\lambda_2 = 1, x = \tfrac{1}{2}(\lambda_2^2 - 1)\mu_2 = 0$. This is impossible.

If $\lambda_2 = 2, \lambda_5 - \lambda_5^{-1} = 9/2$ *i.e.* $(\lambda_5 - 9/4)^2 = 97/16$ and λ_5 is irrational. This is impossible. Similarly $\lambda_2 = 3$ is impossible.

If $\lambda_2 > 3$, take integral parts of both sides of (7), $\lambda_5 = 3\lambda_2$ and by (7) $\lambda_5^{-1} = 3\lambda_2^{-1}$. Thus $3\lambda_2 = \tfrac{1}{3}\lambda_2$. This is impossible.

The original assumption that a triangle of the required type exists is false and the conjecture is true.

The result may be stated as " there is no triangle the lengths of whose sides, medians and one perpendicular are rational numbers ". Other results of this type may be established by similar arguments. H. G. EGGLESTON.

2205. *On conics which touch five given conics.*

In connection with Mr. Robson's article in *Gazette*, 300, it may be found simpler, for the purpose of tabulating the results, to use the notation $[p, q, r]$ to mean the number of conics which pass through p given points and touch q given lines and r given conics, where $p + q + r = 5$. We now do not need the

functions ξ, η, ζ. This brings out more clearly the fact that (in the original notation)

$$\xi[p-1, q, r] = \eta[p, q-1, r] = \zeta[p, q, r-1],$$

each of which is equal to $[p, q, r]$ in the new notation.

The formula

$$\zeta[p, q, r] = 2\xi[p, q, r] + 2\eta[p, q, r],$$

neatly obtained by Mr. Robson, would become

$$[p, q, r+1] = 2[p+1, q, r] + 2[p, q+1, r].$$

By the principle of duality, $[p, q, r] = [q, p, r]$.

The values of the function $[p, q, r]$ may be tabulated thus :

1		2		4		4		2		1
	6		12		16		12		6	
		36		56		56		36		
			184		224		184			
				816		816				
					3264					

where the rows are for $r=0$ to $r=5$, and the terms in each row are for $p=0$ to $p=(5-r)$. A term in any row after the first is obtained by adding the terms on either side of it in the row above and multiplying by 2.

We may even write

$$[p, q, r] = 2^r \left\{ 2^a + \binom{r}{1} 2^b + \binom{r}{2} 2^c + \dots + \binom{r}{r} 2^k \right\},$$

where $a = \min(p+r, q)$, $b = \min(p+r-1, q+1)$, ... $k = \min(p, q+r)$, but the advantages of this formula are doubtful. E. J. F. Primrose.

2206. *The sphere theorem in hydrodynamics.*

P. Weiss in *Proc. Cambridge Phil. Soc.*, 40, (1945) discussed a certain function and showed that it had all the properties required for it to be the perturbation potential due to a sphere placed at the origin in an infinite liquid in irrotational motion. His work later appeared in textbooks on hydrodynamics. The following is a direct method of obtaining this function from first principles.

Let $\phi(x, y, z)$ be the potential of the liquid, which is in irrotational motion, before the sphere is placed at the origin. Then $\phi(x, y, z)$ is a harmonic function whose only singularities are outside $r=a$, where a is the radius of the sphere. Hence ϕ has an expansion in solid harmonics given by

$$\phi = \sum_{n=0}^{\infty} r^n S_n,$$

valid in $r \leqslant a$, where $S_n \equiv S_n(\theta, \omega)$ is a surface harmonic of degree n, using r, θ, ω as spherical polar coordinates.

Hence

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \sum_{0}^{\infty} n a^{n-1} S_n.$$

The inverse point of (r, θ, ω) with regard to the sphere $r=a$ is (R, θ, ω) where $Rr = a^2$. If $\psi(x, y, z)$ is the perturbation potential due to this sphere placed at the origin, then ψ must satisfy the following conditions :

(*a*) ψ is harmonic and regular at every point on and outside $r=a$;

(*b*) $\psi = 0$ at $r = \infty$;

(*c*) $\partial\psi/\partial r = -\partial\phi/\partial r$ on $r=a$.

Thus ψ may be expanded in the form

$$\psi = \sum_{n=0}^{\infty} r^{-n-1} S_n',$$

where S_n' is a surface harmonic of degree n. Condition (c) gives

$$\sum_0^{\infty} (n+1)a^{-n-2} S_n' = \sum_0^{\infty} n a^{n-1} S_n,$$

for all θ, ω in $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \omega \leqslant 2\pi$.
Thus

$$(n+1)S_n' = n a^{2n+1} S_n.$$

Hence

$$\begin{aligned}\psi(x, y, z) &= \frac{a}{r} \sum_0^{\infty} \frac{n}{n+1} R^n S_n \\ &= \frac{R}{a}\left\{\sum_0^{\infty} R^n S_n - \sum_0^{\infty} \frac{R^n}{n+1} S_n\right\} \\ &= \frac{R}{a}\left\{\sum_0^{\infty} R^n S_n - \int_0^1 \left[\sum_0^{\infty} t^n R^n S_n\right] dt\right\}.\end{aligned}$$

Now $\phi(Xt, Yt, Zt) = \sum\limits_0^{\infty} t^n R^n S_n$, where (X, Y, Z) is the inverse point of (x, y, z).

Thus
$$\psi(x, y, z) = \frac{R}{a}\left\{\phi(X, Y, Z) - \int_0^1 \phi(Xt, Yt, Zt)\, dt\right\}$$

where $X = a^2x/r^2$, $Y = a^2y/r^2$, $Z = a^2z/r^2$.

The form of ψ given above seems more convenient to remember than that usually given. G. POWER and A. I. MARTIN.

2207. *Mean value theorems for higher derivatives.*

The well-known theorem that if $f(t)$ possesses a second derivative, then there is a number ξ between $x - h$ and $x + h$ such that

$$\{f(x+h) - 2f(x) + f(x-h)\}/h^2 = f''(\xi)$$

may be proved by considering the auxiliary function

$$\phi(t) = f(t) - At - Bt^2.$$

For A and B may be chosen so that $\phi(x-h) = \phi(x) = \phi(x+h)$, and then, by Rolle's theorem, there exist numbers ξ_1 and ξ_2, $x - h < \xi_1 < x < \xi < x + h$ (supposing h positive), such that

$$\phi'(\xi_1) = \phi'(\xi_2) = 0.$$

Hence, again by Rolle's theorem, there exists a number, ξ, $\xi_1 < \xi < \xi_2$, such that $\phi''(\xi) = 0$. But $\phi''(\xi) = f''(\xi) - 2B$, and from the defining equations for A and B we easily find that

$$2h^2 B = f(x+h) - 2f(x) + f(x-h),$$

which establishes the result.

By considering

$$\phi(t) = f(t) - A_1 t - A_2 t^2 - \dots - A_n t^n$$

in the same way, the corresponding result for the nth derivative may be obtained. The coefficient A_n is, of course, the only one that need be determined.

Although this process is surely well-known it does not appear in the ordinary texts: it seems simpler than that given by Hardy, *Pure Mathematics*, (9th edition, p. 330, Exx. LXVII, No. 1), where the argument is more in the nature of a verification. D. A. T. WALLACE.

2208. *On prime triangles.*

Arising from Mr. Price's article in the *Gazette*, May 1949, p. 121, the following proposition may interest readers.

Let a number which can be assigned to the hypotenuse of a prime right-angled triangle be called a hypotenuse number. Then the product of any two hypotenuse numbers is a hypotenuse number. (For instance, $5 \times 13 = 65$, $5 \times 17 = 85$, $5 \times 41 = 205$.)

The proposition is true, not trivial and not particularly easy to prove.

W. D. EVANS.

2209. *Simple Equations.*

In the report published by the Mathematical Association in 1934 on the teaching of algebra, it is pointed out that the use of the equality sign needs care. "The common error of replacing $\therefore$ by $=$ should be carefully avoided since it involves giving two different meanings to the sign $=$. The correct use of this sign is to assert that two numbers are equal: the misuse asserts that one statement is logically equivalent to another" (*Report*, p. 21).

It can be urged, however, that in solving equations and elsewhere in mathematics, one *does* often want to assert that a statement is logically equivalent to another, and that $\therefore$ is not the sign of logical equivalence. The sign used in *Principia Mathematica* is $\equiv$; but $\longleftrightarrow$ is also used by modern writers on logic, and this is a very suggestive sign especially when it is used in conjunction with the sign $\longrightarrow$ for "implies".

The beginner who writes

$$\begin{aligned} 7x - 14 &= 3x - 2 \\ = \qquad 4x &= 12 \\ = \qquad x &= 3 \end{aligned}$$

should be told that he has used the wrong sign for logical equivalence, but ought he not also to be told what is the correct sign? If he changes two of his equality signs into equivalence signs, his solution becomes correct.

The use of checks is rightly encouraged in the same report. On p. 23, it says: "Most boys, even very intelligent ones, seem to have an irresistible temptation to use the following arrangement" for checking a solution $x = 4$ found for the equation $\frac{1}{3}(x - 1) + 4 = 9 - \frac{2}{5}(3x - 2)$:

$$\begin{aligned} \tfrac{1}{3}(4 - 1) + 4 &= 9 - \tfrac{2}{5}(12 - 2) \\ \therefore \qquad 1 + 4 &= 9 - 4 \\ \therefore \qquad 5 &= 5. \end{aligned}$$

Whose fault is it that very intelligent boys use this arrangement? Might they not have correctly used $\longleftrightarrow$ instead of $\therefore$ if they had been introduced to that symbol?

A check is sometimes a logical necessity and sometimes only a prudent way of detecting a blunder. If a solution is set out in the form

$$\begin{aligned} 7x - 14 &= 3x - 2 \\ \longrightarrow \qquad 4x &= 12 \\ \longrightarrow \qquad x &= 3 \end{aligned}$$

the check is a logical necessity (and so it is when $\therefore$ is used instead of $\longrightarrow$). Perhaps this is the best layout for beginners. The check can be written

$$x=3 \longrightarrow 7x-14=21-14=7$$
$$x=3 \longrightarrow 3x-\ \ 2=\ \ 9-\ \ 2=7\ ;$$

but it is also perfectly logical to write it in the form

$$\begin{aligned} 7.3-14&=3.3-2 \\ \longleftrightarrow\ 21-14&=9-2 \\ \longleftrightarrow\ \quad 7&=7. \end{aligned}$$

The meanings of $\therefore$ and $\longrightarrow$ are not the same. If S and T are statements, $S\longrightarrow T$ means not -S or T, but " $S\ \therefore\ T$ " asserts S as well as asserting that T follows from it.

An advantage of using $\longrightarrow$ instead of $\therefore$ is that it is not then necessary to begin by using the symbol $=$ as an expression of hope, (*Report*, p. 23).

For example, with the usual meaning of $\sqrt{x}$, the statement

$$(\sqrt{x}=-1)\longrightarrow x=1$$

is correct by the definition of $S\longrightarrow T$; but in

$$\sqrt{x}=-1$$
$$\therefore\ \ x=1$$

the first statement is not true for any value of x.

It is not only for the solution of equations, and indeed not only in mathematics, that the ideas involved in $S\longrightarrow T$ and $S\longleftrightarrow T$ are valuable. A. R.

2210. *On note* 2053 (*Normals to a parabola*).

The results obtained by Mr. McCarthy in his note in the *Mathematical Gazette* 33 (1949), No. 304, can all be obtained much more shortly by the method given below. It is, or ought to be, a well-known method.

1. If the normals to the parabola $x=at^2$, $y=2at$ at the points with parameters t_1, t_2, t_3 concur in the point (α, β), then t_1, t_2, t_3 are roots of the equation

$$at^3+(2a-\alpha)t-\beta=0. \quad \text{.........(1)}$$

Hence

$$t_1+t_2+t_3=0, \quad \text{.........(2)}$$

$$2a-\alpha=a\,\Sigma t_1t_2=-\tfrac{1}{2}a\,\Sigma t_1^2, \quad \text{.........(3)}$$

by (2), and

$$\beta=at_1t_2t_3. \quad \text{.........(4)}$$

Condition (2) is therefore necessary for the normals to be concurrent and it is easy to see that it is also sufficient. The point of concurrence is, by (3) and (4),

$$\alpha=2a+\tfrac{1}{2}a\,\Sigma t_1^2, \quad \beta=at_1t_2t_3. \quad \text{.........(5)}$$

2. The circle through the origin

$$x^2+y^2+2gx+2fy=0$$

meets the parabola where

$$a^2t^4+4a^2t^2+2agt^2+4aft=0,$$

and therefore it meets the parabola at the origin and the points t_1, t_2, t_3 if and only if the latter are the roots of

$$at^3+(4a+2g)t+4f=0.$$

Hence $$t_1 + t_2 + t_3 = 0, \qquad (6)$$
$$4a + 2g = a\,\Sigma t_1 t_2 = -\tfrac{1}{2} a\,\Sigma t_1^2, \qquad (7)$$
$$4f = -a t_1 t_2 t_3. \qquad (8)$$

From (2) and (6) it follows that the feet of concurrent normals lie on a circle through the vertex, and by (7) and (8) we see that the circle is

$$2(x^2 + y^2) - ax(8 + \Sigma t_1^2) - a y t_1 t_2 t_3 = 0,$$

or, by (5),

$$2(x^2 + y^2) - 2x(2a + \alpha) - \beta y = 0.$$

IDA W. BUSBRIDGE.

2211. *On Note* 2058 : *a property of the cyclic quadrilateral.*

If the opposite sides AB, $B'A'$ of the cyclic quadrilateral $ABA'B'$ meet at C' and the opposite sides BA', AB' meet at C, then the bisectors of the angles $AC'B'$, BCA meet at right angles at a point G which lies on the line joining the midpoints of the diagonals. Assume that the sides meet when produced in the directions indicated, so that $C'G$ meets BA' at L between C' and G.

$$\angle GLC = BLC' = ABC - \tfrac{1}{2}BC'A' = ABC - \tfrac{1}{2}(ABC - BA'C')$$
$$= ABC - \tfrac{1}{2}(ABC - BAC) = \tfrac{1}{2}(ABC + BAC)$$
$$= \tfrac{1}{2}(180^\circ - ACB) = 90^\circ - GCL.$$

Thus $\angle CGC' = 90^\circ$.

The triangles CBB', CAA' are similar. Therefore if M is the midpoint of BB' and N is the midpoint of AA', the triangles CBM, CAN are similar, since $\angle CBM = \angle CAN$ and the sides about these angles are proportional.

Thus $\qquad CM : CN = CB : CA$, and $\angle BCM = \angle ACN$.

Hence $\qquad \angle GCM = \angle GCN$,

and CG divides MN in the ratio $CB : CA$.

Similarly $C'G$ divides MN in the ratio $C'B' : C'A$.

These ratios can be proved equal either by the use of the formula

$$a/\sin A = b/\sin B,$$

or by the use of what is practically a converse of Euclid, Book VI, Prop. 7 (Hall and Stevens, *Euclid* (1898), p. 321). J. J. WELCH.

2212. *The shortest distance between two skew lines.*

The aim of this note is to show that there exists a unique shortest distance between two skew lines, and that this shortest distance is their common perpendicular. The proof is entirely algebraic, avoiding any appeal to theorems of euclidean solid geometry on the one hand, or to the calculus on the other.

Let the two skew lines be

$$L\ : (x-a)/l \;\; = (y-b)/m \;\; = (z-c)/n,$$
$$L' : (x-a')/l' = (y-b')/m' = (z-c')/n'.$$

It is required to prove that there exist unique points P, P' of L, L' respectively, such that the distance PP' is a minimum, and that PP' is then perpendicular to both L and L'.

Lemma. If $A, B, \dots, H$ denote the co-factors of $a, b, \dots, h$ in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

and if $a>0$, $C>0$, then

$$ax^2+2hxy+by^2+2gx+2fy+c \geqslant \Delta/C,$$

with equality if and only if $x=G/C$, $y=F/C$.

For, $\quad ax^2+2hxy+by^2+2gx+2fy+c$

$$=(ax+hy+g)^2/a+\{a(hx+by+f)-h(ax+hy+g)\}^2/Ca+\Delta/C \geqslant \Delta/C,$$

with equality if and only if

$$ax+hy+g=0=hx+by+f,$$

that is, if and only if $x=G/C$, $y=F/C$.

Proof. Let $P=(a+lu,\ b+mu,\ c+nu)$, $P'=(a'+l'u',\ b'+m'u',\ c'+n'u')$ be general points of L, L' respectively. Then

$$PP'^2=\Sigma(lu-l'u'+a-a')^2$$
$$=u^2\Sigma l^2-2uu'\Sigma ll'+u'^2\Sigma l'^2+2u\Sigma l(a-a')-2u'\Sigma l'(a-a')+\Sigma(a-a')^2.$$

Now $\Sigma l^2>0$, and $(\Sigma l^2)(\Sigma l'^2)-(\Sigma ll')^2>0$, since the lines are skew. We can therefore apply the lemma and obtain

$$PP'^2 \geqslant \begin{vmatrix} \Sigma l^2 & -\Sigma ll' & \Sigma l(a-a') \\ -\Sigma ll' & \Sigma l'^2 & -\Sigma l'(a-a') \\ \Sigma l(a-a') & -\Sigma l'(a-a') & \Sigma(a-a')^2 \end{vmatrix} \Big/ \{(\Sigma l^2)(\Sigma l'^2)-(\Sigma ll')^2\}$$

$$= \begin{vmatrix} l & m & n \\ -l' & -m' & -n' \\ a-a' & b-b' & c-c' \end{vmatrix}^2 \Big/ \{(\Sigma l^2)(\Sigma l'^2)-(\Sigma ll')^2\} \qquad \text{.........(i)}$$

There is equality if and only if

$$u\Sigma l^2-u'\Sigma ll'+\Sigma l(a-a')=0 \qquad \text{.........(ii)}$$
$$-u\Sigma ll'+u'\Sigma l'^2-\Sigma l'(a-a')=0, \qquad \text{.........(iii)}$$

and these equations can be solved to give unique values for u and u', since

$$(\Sigma l^2)(\Sigma l'^2)-(\Sigma ll')^2 \neq 0.$$

Further, equations (ii) and (iii) are the conditions that PP' should be perpendicular to L and to L' respectively, while equation (i) yields the usual formula for the shortest distance between two skew lines, as given, for instance, in Bell, *Coordinate Solid Geometry*, p. 58. L. E. CLARKE.

2213. *On the differentiation of a function of a function.*

The importance of the method of differentials and, in particular, of the fundamental theorem on the permanence of the first differential has been stressed by Mr. E. G. Phillips* in a recent article, reference being made to functions of more than one variable. It is the object of this note to point out that, in the case of functions of one variable, a very simple and complete† proof of the theorem on differentiation of a function of a function can be obtained by using the same method of proof as that usually given for the theorem on the permanence of the first differential.‡ It is not, in fact, necessary to introduce the definition of a differential, though it is the method of differentials rather than that of derivatives § which is used.

* *Mathematical Gazette*, XXXIII, 202.

† It was pointed out by Carslaw (*Bull. American Math. Soc.*, vol. 29) in 1923 that proofs given in standard English books of that date were not complete. For references to complete proofs in earlier foreign books, see Carslaw, *loc. cit.*

‡ See, for instance, E. G. Phillips, *A Course of Analysis*, p. 229.

§ For a complete proof using the method of derivatives, see G. H. Hardy, *Pure Mathematics* (4th and later editions).

To obtain the theorem, we suppose that u is a differentiable function of x and that y is a differentiable function of u. Then *

$$\Delta u = \left(\frac{du}{dx} + \epsilon_1\right) \Delta x, \quad \text{.............(1)}$$

$$\Delta y = \left(\frac{dy}{du} + \epsilon_2\right) \Delta u, \quad \text{.............(2)}$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$ and $\epsilon_2 \to 0$ as $\Delta u \to 0$, hence as $\Delta x \to 0$. Substituting for Δu from (1) in (2), we find that Δy is given by an expression of the form

$$\Delta y = \left(\frac{dy}{du}\frac{du}{dx} + \epsilon\right) \Delta x,$$

where $\epsilon \to 0$ as $\Delta x \to 0$. Hence y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

It should perhaps be emphasized that (2) holds both for zero and non-zero values of Δu, so that the proof given above is complete. On the other hand, it is only legitimate to write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}. \quad \text{.............(3)}$$

and let $\Delta x \to 0$ if † Δu is different from zero for all sufficiently small non-zero values of Δx.

W. L. C. SARGENT.

2214. *Differential operators in Leibniz's theorem and integration by parts.*

A well-known proof of Leibniz's theorem on the nth derivative of a product makes use of the ideas of operational methods in a somewhat special way. The operation of differentiation (D) applied to the product (uv) of two functions is split up into the operations (D_1 and D_2) of differentiation applied to the separate functions (u and v) and their derivatives. Then the binomial theorem is applied to the right-hand side of the identity $D^n = (D_1 + D_2)^n$. In this the index n is a positive integer.

The natural question (asked by a student) is : What if n is not a positive integer?

If n is -1 (or any negative integer) D^n signifies integration (single or repeated) and the answer to the question provides an interesting modification of integration by parts.

Taking first $n = -1$, we have :

$$\begin{aligned} D^{-1}(uv) &= (D_1 + D_2)^{-1}(uv) = (D_1^{-1} - D_1^{-2}D_2 + D_1^{-3}D_2^2 - \ldots)(uv) \\ &= vD^{-1}u - DvD^{-2}u + \ldots ; \end{aligned}$$

i.e.
$$\int uv\,dx = v\int u\,dx - \frac{dv}{dx}\iint u\,dx\,dx + \ldots ,$$

which is integration by parts continued indefinitely.

* The notation used is that found in Phillips, *A Course of Analysis*. The proof is essentially the same as that given in Verblunsky, *Functions of a Real Variable*, p. 64, but the difference in notation brings out the analogy with the proof of the theorem on the permanence of the first differential. For similar proofs, see also Chaundy, *The Differential Calculus*, p. 71 and Gibson, *Advanced Calculus*, p. 66.

† Cf. Durell and Robson, *Elementary Calculus*, Vol. 1, Phillips, *A Course of Analysis*, and Stewart, *Advanced Calculus*, where it is pointed out that the proof using (3) only holds if this restriction is made.

If we stop the series after the first term, so that

$$(D_1 + D_2)^{-1} = D_1^{-1} - (D_1 + D_2)^{-1} D_2 D_1^{-1},$$

we have ordinary integration by parts, viz.

$$\int uv\,dx = v \int u\,dx - \int \left(\frac{dv}{dx} \int u\,dx \right) dx\,;$$

which is recognisable as the usual form,

$$\int U \frac{dV}{dx}\,dx = UV - \int V \frac{dU}{dx}\,dx,$$

if we write $u = dV/dx$, $v = U$, so that $uv = U(dV/dx)$, $\int u\,dx = V$, $dv/dx = dU/dx$.

For other negative integral values of n the method leads to a systematic and simple way of evaluating repeated integrals of products of two functions of sufficiently simple types—particularly when one is a fairly low positive integral power of x and the other is amenable to repeated integration.

As a concrete example, to find

$$\text{(i) } \int x^3 \sin x\,dx \quad \text{and} \quad \text{(ii) } \int\!\!\int\!\!\int\!\!\int\!\!\int x^3 \sin x\,dx\,dx\,dx\,dx\,dx.$$

The first $= (D_1 + D_2)\, x^3 \sin x$, where D_1 applies to $\sin x$ and D_2 to x^3,

$$= (D_1^{-1} - D_1^{-2} D_2 + D_1^{-3} D_2^2 - \ldots)\, x^3 \sin x$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x,$$

exactly as in applying integration by parts three times.

In the second the operator D^{-5} becomes

$$(D_1 + D_2)^{-5} = D_1^{-5} - 5 D_1^{-6} D_2 + \frac{5\,.\,6}{1\,.\,2} D_1^{-7} D_2^2 - \frac{5\,.\,6\,.\,7}{1\,.\,2\,.\,3} D_1^{-8} D_2^3 + \ldots$$

and $D_1^{-5}(x^3 \sin x) = x^3 D^{-5} \sin x = -x^3 \cos x$, etc.

Hence (ii)

$$= -\cos x\,.\,x^3 + 5 \sin x\,.\,3x^2 + 15 \cos x\,.\,6x - 35 \sin x\,.\,6$$

$$= -x^3 \cos x + 15x^2 \sin x + 90\,x \cos x - 210 \text{ six } x.$$

In some other cases, *e.g.* in evaluating $\int e^{ax} \cos bx\,dx$, the method leads to infinite series. If these series can be summed algebraically (without regard to convergence) the results will be correct.

C. WALMSLEY.

2215. *Notes on Conics.* 14. *An analytical proof of Pascal's theorem.*

The collector may welcome the following :

The two equations

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right)\left(\frac{x}{c} + \frac{y}{d} - 1\right) = \lambda xy, \quad \frac{x}{a} + \frac{y}{b} - 1 = \lambda cd,$$

together give

$$\left(\frac{x}{c} - 1\right)\left(\frac{y}{d} - 1\right) = 0.$$

This proves that if a hexagon in a conic has two pairs of parallel sides, the third pair of sides is parallel, and as Pascal himself said, this special case establishes the general theorem. Needless to say, my axes are oblique.

E. H. N.

REVIEWS.

Advanced Plane Geometry. By C. Zwikker. Pp. xii, 299. 20 fl. 1950. (North-Holland Publishing Co., Amsterdam)

Mr. Zwikker remarks that plane geometry has to be collected from three types of textbook: from geometry books we get the conics, and often not much else; problems requiring derivatives and integrals are treated in the calculus books; and non-algebraic curves occur mainly in books on kinematics. He has therefore attempted to gather the most important properties of plane curves into one volume, to treat them by a uniform method, and to indicate applications to physical problems. His position as Technical Director of the Light Division of Philips of Eindhoven accounts for his frequent references to circuit problems, to electromagnetic theory and, in general, to two-dimensional potential problems; and since he is professionally interested in electro-technics it is not surprising that the uniform method he has chosen is that of the complex variable, in which a point is a complex number $z=x+iy$. In fact, the book is an essay on the applications of the algebra of the complex domain to the study of plane curves. The method, though well enough known, has not been too generously treated in elementary books, and no doubt many teachers will be surprised by the neatness and elegance of some of the proofs of familiar geometrical properties; on the other hand, conformity to one method is almost certain to entail an occasional clumsiness or forcing. It is a measure of Mr. Zwikker's success that he rarely makes us feel that he is cracking his nuts with a steam-hammer, and those of us (the old fogeys, our young geometers would say) who can find pleasure in a neat proof of some odd property of the cardioid or the strophoid, and who like that kind of geometry in which an occasional figure is permissible, will recapture with Mr. Zwikker's aid some of the enjoyment we derived in our youth from the study of the chapter on "Well-known Curves" in whatever calculus textbook we cherished in those days.

There is, however, nothing antiquated about Mr. Zwikker's book. He has an eye for the "neat solution", but his main design is to keep geometry close to those applications, chiefly in the electrical field, which can be made. The technique of the calculus is freely used, and in dealing with cubics the Weierstrassian elliptic function $\wp(z)$ plays the main part; previous knowledge of its properties is not assumed, but would help the reader.

A short historical survey of the literature is added; it is somewhat surprising to find no mention of Frank Morley and F. V. Morley, *Inversive Geometry*, a fascinating volume which provides a much stronger theoretical basis for this kind of geometry as well as an extensive survey of geometrical applications.

Mr. Zwikker has chosen to write in English, and he is almost always clear and precise, if occasionally a little stilted. The printing is good, the diagrams (nearly 300) excellent. A brief list of chapter topics must serve to indicate the extent of the ground covered: the plane geometrical interpretation of complex algebra; straight line; triangle; circle; algebraic curves; ellipse; hyperbola; parabola; involutes and evolutes; pedals; areas and other integrals; envelopes; orthogonal trajectories; kinked curves; spirals; lemniscate; cycloid; epi- and hypocycloid; cardioid and limaçon; gear wheel tooth profiles. T. A. A. B.

Probability and the Weighing of Evidence. By I. J. Good. Pp. viii, 119. 16s. 1950. (Griffin)

"The aim of the present work is to provide a consistent theory of probability that is mathematically simple, logically sound and adequate as a basis for scientific induction, for statistics, and for ordinary reasoning." This

unobjectionable programme is fittingly introduced by a Chapter " Theories of Probability ", which contains a classification of earlier systems, from Venn to Mises, Jeffreys, Ramsey, Koopman and others. Chapter 2, " The Origin of the Axioms," proposes " to show that the axioms stated in Chapter 3 are not chosen in a haphazard manner ". There are six of them, and in order to apply them to judgments concerning degrees of belief, we must have " rules ". The first of them states that an expression $P(E \mid H)$ is a number subject to the axioms and, at the same time, a reasonable belief in E, when H is assumed. Deductions may be drawn by using the axioms together with the body of beliefs, whereby the latter is enlarged ; if this procedure leads to contradictions, then the body of beliefs is called " unreasonable ".

Next, we have " suggestions " for forming bodies of belief. They contain such advice as that " in order to avoid ultimate contradictions all probability judgments should be honestly held, and should be arrived at unemotionally ". The author contends that " the trichotomy into axioms, rules and suggestions is perhaps the ideal form for any scientific theory " (p. 31).

In Chapter 6, " Weighing Evidence," the author introduces the concept of " factor in favour of H ", which is the ratio of the odds in favour of a hypothesis after the experiment to those before it, and is numerically equal to the " likelihood ratio " $P(E \mid H)/P(E \mid \bar{H})$, where $\bar{H}$ is the negation of H. The logarithm of the likelihood ratio is called " weight of evidence " and that of the odds is called " plausibility ". These notions are quite useful and appropriate, for instance, to explain the essence of a sequential probability ratio test. For those who wish to practise, there are some exercises (one of them concerning evidence, surely inadmissible in a court of law, which gives odds of 386 : 1 in favour of a suspicion which the Index, though not the text, describes bluntly as " adultery, p. 74 ").

The reviewer hopes that he has given a fair summary of the theory presented in this book, but he admits that he had some difficulty in sorting it all out. This is due to a curious balance between the main text, footnotes (at least one on nearly every page) and the Index. Thus a statement on p. 80 about the " most probable value " makes sense only when one has found one's way *via* a footnote to the Index, which gives the relevant definition. Again, the Index contains an elaborate statement on " implication ", after the author has disclaimed any attempt at deciding between various opinions concerning this, one should think fundamental, concept. Moreover, the author suffers from a conspicuous aversion to precision. How else could one explain such a trite statement as that on p. 106 : " a theorem due to Abel (and published in his collected works) " or his repeated reference to a theorem by Borel which, according to the Index(!) should " perhaps better be called the Borel-Cantelli theorem " ?

The last chapter, " Statistics and Probability," deals with systems of statistics, not quite satisfactorily. Many remarks are made in a cavalier fashion, as for instance on p. 87. " There are reasons," says a footnote, for using an estimate of the variance different from that given by the maximum likelihood method. There are indeed ; why not mention them? It is hardly " natural " that the expected number of observations in a sequential test is smaller than in an equivalent test with fixed sample size, unless everything that follows from assumptions is thereby natural. The sentence on p. 86 : " Another reason for preferring the simpler curve is that any given simple curve is found in practice to occur, as an approximation, more often than any given complicated curve " is, at the best, a plea to perpetuate a usage without inquiring why it has ever been started. The formula for defining confidence intervals on p. 101 can not be understood by anyone who does not already know all about the theory of estimation by intervals. One takes also

exception to the suggestion that " every example should be treated on its merits, unless the statistician is short of time " (p. 98/9).

There are very few misprints in this book, but something has gone wrong in the formula on p. 55 with an undefined x_1', and all references in the Index to pp. 104 and 105 were inserted when it was not yet known that page 104 would, in fact, be blank.

To sum up, then, the book contains much that is irritating and much that is stimulating. It should be perused by all those interested in more recent developments of the theory of probability, together with such contributions as, for instance, Barnard's paper to the Royal Statistical Society in March 1949. It contains also much that is written in a lighter vein, and the reviewer would like to quote the following sentence : " Use enough common sense to know when ordinary common sense does not apply " (p. 77). S. V.

New Junior Mathematics. By J. J. DE KOCK and A. J. VAN ZYL. Second edition. Pp. xii, 392. 1949. (Maskew Miller, Limited, Cape Town)

This book is written for secondary schools in South Africa, and treats separately Arithmetic, Geometry, Algebra and Trigonometry. This makes reference easy, but the subjects are by no means in watertight compartments ; on the contrary, there is continuous application from one section to another, and the book reads as a continuous course in mathematical thinking and method.

The teaching is done not by direct instruction, but by multitudinous questions, oral and written, by means of which the pupil instructs himself, and subsequent summaries gather together what has been learnt. If there is a fault here it might be that the summaries are not always in the correct place, *e.g.* the angle sum of a triangle is found in an example on p. 63, is quoted in bookwork on p. 80, but is not stated in bookwork till p. 110. But this is exceptional, and the methods used to help pupils to discover all essential facts are most ingenious. Much use is made of questions in which only one word is to be fitted in by the pupil, or in which he has to chose between several alternatives. There are frequent vocabulary tests to ensure that a pupil understands and remembers the various terms used.

In scope the book may be said to cover the first two years of a secondary school course here. There is little Arithmetic, the emphasis being on correct use of numbers, significant figures, and statistical graphs. Logarithms are not used. The Geometry course is practical, with work on angles leading to triangles as the basic figure, treated fully with congruence and similarity, and leading to other figures. There is interesting work on navigation, surveying, and frequent historical references. This leads to the last chapter of this section on " Reasoning in Geometry ", which contains essential theorems. Angle properties of the circle are included, but not the rectangle properties of chords, though all methods leading up to them have been included. Algebra is based on formula and problem, leading to equation. Pure manipulation is introduced only as necessary, and hence the work on equations, factors, fractions comes towards the end. Graphs of linear functions are dealt with thoroughly, and provide the first method of solution of simultaneous equations. Quadratic functions are not introduced either in graphs or in equations. Some teachers might here complain of a shortage of drill exercises, and so some extra sets are inserted at the end. The last section of the book is on trigonometry, with one chapter on the tangent, and one on the sine and cosine.

The authors lay stress on the need for frequent revision to go over again what has been done. Hence instead of a large number of examples in any one section they give 68 revision papers through the book. There is at the

beginning a good section addressed to the teacher, emphasising that not too long must be spent on the oral part, of which there is so much, and suggesting that explanations and historical sections can be expanded by the teacher. Any teacher starting mathematics, as opposed to arithmetic alone, will find this book stimulating, and full of ideas. As an example of this, the first set of examples on " Reasoning in Geometry " is on reasoning in everyday life on subjects such as games, birthdays, which are in the minds of children.

There are some imperfections. In the reference section horizontal is defined as " at right angles to the vertical ", and vertical as " at right angles to the horizontal ". On page 386 it is said that 6·3, to two significant figures, " can vary from 6·25 to 6·34 ".

The diagrams and printing are clear, and care has been given to pagination. The book finishes with tables, including three-figure trigonometrical tables, and a reference section, repeating definitions as well as referring to the section in the book, and an index. K. S. S.

Mathematics Dictionary. By G. James and R. C. James. Revised edition. Pp. vi, 432. 56s. 1949. (D. Van Nostrand ; Macmillan)

This book is a substantially enlarged version of that bearing the same title, introduced by the authors in 1943, and its publication is thus a testimony to the well-deserved popularity of the original work.

All the entries appearing in the former version have been incorporated in the one under review, but there are now, in addition, several hundred of a non-elementary character, covering a wide range of subjects from point-set topology and group-theory to analytical mechanics and potential functions. The appendix of tables and formulae is also much more comprehensive, and now includes, *inter alia*, a list of the symbols used in mathematical logic and the theory of sets of points.

The book is well printed and the general layout is admirable, and should prove a useful addition to any mathematical library. J. H. P.

The Meaning of Relativity. By Albert Einstein. Fourth edition. Pp. v, 145. 7s. 6d. 1950. (Methuen)

This edition of Professor Einstein's well-known little book is a reprint of the third edition published in 1946 with the addition of the much-heralded " Appendix II, Generalized Theory of Gravitation ". The present notice may, therefore, be confined to this new material. It must be unique in modern times for a distinguished author to choose this method of publishing what he is reported to regard as a major contribution resulting from many years' thought on the subject.

The form is no less surprising than the method of publication. The Appendix consists of fifteen tersely written pages, consisting almost entirely of mathematics with only the barest indication of its physical interpretation. Apart from a very brief introductory statement, there is no assessment of the significance or advantages accruing.

More than twenty years ago, Einstein himself initiated a fresh series of attempts to establish a unified field-theory of gravitation and electromagnetism. The reasons for demanding such a theory are well known and need only the briefest recapitulation. They are that the four-dimensional Riemannian geometry (characterised by a " fundamental " or " metric " tensor g_{ik}) employed in general relativity to describe the behaviour of space-time and matter contains no features that can be used fully to describe an electromagnetic field. Therefore, while general relativity appears to give a complete account of gravitation, it includes no treatment of electromagnetism. More-

over, even its treatment of gravitation is not really complete. For in its fundamental field-relations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik} \quad\text{.............................(1)}$$

the left-hand side depends only on the geometry of the Riemannian space. But the stress-energy tensor on the right is taken to include, for instance, the energy of any electromagnetic field which is present, despite the fact that the theory has no place for the equations of the electromagnetic field itself. Thus it is clear why Einstein regards his original theory as " only a provisional treatment of the total problem ".

If a unified field-theory is possible, then the field must require for its specification something more than a fundamental symmetric tensor g_{ik} in four dimensions. Now the trouble is not to find *something* more general ; it is that *too many* apparent possibilities present themselves. As is well known, a large number have been proposed and investigated. Hitherto none has had any pronounced advantage over the others on grounds of simplicity nor of progress in physical insight, and none has made any new verifiable physical prediction.

Judged by the criteria which have just been indicated, Einstein's new suggestion does appear to possess some advantage in the simplicity of the idea from which it starts. Whether it may prove to possess advantages in the other respects it is too early to say, and Einstein's own treatment stops short of any possibility of applying such tests.

Einstein's suggestion is still to base the theory on a single fundamental tensor g_{ik} and merely *to drop the requirement that it be symmetric* (which means that g_{ik} should have 16 instead of 10 independent components). As he hastens to point out, the symmetric and skew symmetric parts of g_{ik} are then separate tensors ; so at first sight nothing seems to be achieved beyond the arbitrary annexation of an unrelated skew tensor. But this objection is met by the fact that Einstein then proposes to use the *entire* new g_{ik} to define the correlated contravariant tensor g^{ik} according to the usual equation

$$g_{is}g^{it} = \delta^t{}_s = g_{si}g^{ti}.$$

This results in the new fundamental tensor playing a part as an irreducible entity in all that follows.

Einstein then writes down the standard equations

$$\delta A^i = -\Gamma^i_{st}A^s dx^t$$

for the parallel displacement of a vector (*i.e.* for an " affine connexion "), where the parameters Γ^l_{ik} are to be related in some suitable way to the g_{ik}. These parameters take the place of the usual Christoffel symbols, but will no longer be symmetric in the suffixes i, k. So Einstein has to point out that, while they lead in the usual way to the definition of absolute derivatives of a vector, they produce three sorts of such derivatives according as Γ^l_{ik}, or its transpose, or its symmetric part is inserted in the definition. The definition is extended also to obtain absolute derivatives of tensor densities.

The relations between the g_{ik} and Γ^l_{ik} adopted by Einstein are those given by the vanishing of one set of the absolute derivatives of the g_{ik}. These relations are formally the same as those encountered at one stage in the standard treatment of the symmetric field, though, owing to the absence of symmetry, they do not lead to the usual Christoffel symbols.*

* The treatment given by P. G. Bergmann *Introduction to the theory of Relativity* (New York, 1942), 67-74, is particularly useful for comparison since it makes very explicit the reasons for, and consequence of, the symmetry of the Γ^l_{ik} in the standard theory.

A curvature tensor is then defined in terms of the Γ^l_{ik} in the same way as the Riemann-Christoffel curvature tensor is defined in terms of the Christoffel symbols. The contraction of the new tensor in various ways yields several distinct tensors of order two, in place of the single (Ricci or Einstein) tensor R_{ik} which appears in (1). One particular combination U_{ik} of these resulting tensors is then isolated, apparently because of a certain type of auxiliary invariance which it is found to possess.

Now it is well known that Einstein's original gravitational field equations are derivable from a variational principle involving the tensor R_{ik}. He now derives new field equations by substituting U_{ik} for R_{ik} in this principle.

The new equations, as they first appear, are highly complicated. One degree of simplification is achieved by adjoining four further conditions on the Γ^i_{ik} which is shown to be permissible by virtue of the above-mentioned invariance. A second degree of simplification results from adjoining the conditions that the absolute derivatives of $[\det(g_{ik})]^{\frac{1}{2}}$ should vanish.

The final field equations consist of four sets composed of 16, 64, 4, 4 members, each set being expressed by the vanishing of a single symbol in the notation here developed by Einstein. (Certain identities reduce somewhat the number of independent equations.) He offers no further discussion beyond the bare assertion that two resulting identities express the vanishing of the magnetic current density and the conservation of electricity. The particular identities clearly possess the form required for these interpretations.

Einstein describes his results as a system of " utmost simplicity ", and goes on to assert that they " appear to me the natural extension of the general theory of relalivity ".

The insight of such an authority as Einstein must command our respect. Certainly, too, the simplicity of his proposal to start with an unrestricted fundamental tensor makes a favourable appeal. Nevertheless, what has been written here shows how much of the subsequent formulation appears to be entirely arbitrary and how little of it has received physical interpretation. It is clear that a tremendous amount of investigation is required before others than the eminent author himself are enabled to form an opinion of the significance of his work. So far as the reviewer is aware, no such investigation has yet appeared.

W. H. McCrea.

Chance and Choice by Cardpack and Chessboard. Vol. I. By Lancelot Hogben. Pp. 417. 50s. 1950. (Parrish)

As Professor Hogben rightly claims, there are already many useful textbooks which describe the techniques of statistical analysis for the student who is prepared to accept the recurring phrase : " it can be shown that . . . ". There are also available some excellent textbooks for the reader with a good mathematical training. Many of those who have to use statistical methods in their work, however, are not equipped to benefit from the mathematical texts, and yet need more insight into statistical reasoning than is provided by the books of recipes. The purpose of this volume is to provide those whose specialist work lies in the fields of medicine, the biological sciences, agriculture, or sociology, with a detailed exposition, complete as far as it goes, of large sampling theory. All the necessary pure mathematics, from the binomial theorem to beta and gamma functions, is adequately developed as required. If, however, the sub-title " An Introduction to Probability in Practice by Visual Aids " suggests that it is suitable as a first book on statistical method, it misleads. A reader without some previous knowledge of statistical method would lose all perspective.

In conventional terms the main topics discussed in the book are sampling

of attributes, the sampling of variables, significance, the method of moments, correlation, and the analysis of variance, though only three of these six topics might be recognised by a glance at the table of contents. The unfamiliar terminology used by the author arises from his awareness of the semantic difficulties that occur in teaching elementary statistics. Many of its technical terms, *e.g.* normal, association, probability, standard error, unbiassed estimate, carry with them from everyday use connotations that are misleading. Professor Hogben does not hesitate to use less familiar words whenever greater clarity or precision might be achieved thereby. He gives his reasons and chooses well. Thus, for example, for *mathematical probability* he uses *electivity*, in the discussion of the theorem of Bayes the *prior probability* becomes the *commendability* and the *posterior probability* is called the *operational value*. While the advanced student can accept or mentally translate the new terms as he reads, the elementary student grappling with a concept only vaguely understood may be still further confused by giving the concept a new label.

The author's method is patiently to explore the implications of all possible alternatives of any practical procedure. Every step is illustrated by reference to some concrete example which can be numerically calculated and diagrammatically represented. His main thesis is that the correct application of statistical methods depends on the finding of the appropriate model among the apparatus of games of chance, *i.e.* cards, dice, urns, etc. Thus, for example, Chapter 4—The Recognition of a Taxonomic Difference—begins by stating the problem to be solved. " Is vaccination effective against smallpox? " is a question typical of the problem. In terms of a model this becomes the problem of deciding whether there is the same proportion of black balls in two urns containing black and white balls when the information about the contents of the urns is limited to samples. The next step is to consider the implication of the null hypothesis, *i.e.* that the contents of the two urns are identical. A numerical example is discussed and further problems arise. As the proportion of the black balls in the urns is unknown some estimate of it is required. Should we base our estimate on the proportion found in the sample from the first urn? or from the second urn? or should we pool the information obtained from both samples? Each of these possibilities is numerically explored and the results compared. And so the method is continued throughout the chapter, throughout the book. The heavy arithmetical computation and algebraic manipulation which this method entails is never shirked and every step is clearly shown. Some preliminary algebraic results and approximations are established in the first chapter, which in its Pythagorean enthusiasm for figurate numbers is reminiscent of the author's *Visual Algebra*. Thereafter the pure mathematics is provided as required. There is an interesting parallel development of the sampling of attributes with and without replacement so that both the binomial and hypergeometric distributions are obtained and compared, and the Normal distribution is shown to be the limit of both. The discussion of Correlation begins with correlation by ranks, which is later extended to product-moment correlation. Artificial model scoring systems are devised and their results carefully analysed in a very illuminating chapter on the Nature of Concomitant Variation. Enough has been said to indicate that Professor Hogben's treatment of his subject is thorough, original and instructive.

The visual aids call for comment. It is difficult to believe that the readers for whom this book is intended need or are helped by all the beautifully drawn devices. It is not suggested that visual aids are unnecessary or unhelpful, but that some of these particular visual aids are ineffective because they are too elaborate. For example, in some diagrams, *e.g.* Figs. 62, 63, the die

faces are necessarily so small that the score patterns they represent would be much easier to grasp if they were simply replaced by digits. The use of two colours is not always effective ; in the review copy the overprinted red lines of Fig. 55 unhappily are seriously misplaced and the diagram is misleading. The printing of the elaborate two-colour diagrams and charts must represent a large fraction of the high cost of this publication. It is regrettable that some potential readers of this book may be lost because of the expense of these visual aids of doubtful value.

There are two serious defects which can fortunately be remedied in the next edition. The first is that there is no index, a very serious omission in view of the large size of the book and of its unconventional treatment. The second is that the statistical exercises are without answers or comments, an omission which will not encourage the private student.

The book deserves a circle of readers wider than that for which it is specifically written. There is a steadily growing demand for courses of elementary statistics for all whose work involves interpretation of numerical data, but little thought has yet been given to the heuristic difficulties. As the work of an experienced teacher, Professor Hogben's book will be as valuable to those who are teaching as to those who are using statistics, though non-medical teachers will be disappointed to find that the exercises are mostly based on emphatically medical data. The second volume of this valuable work will be awaited with interest.

Finally, the publishers deserve commendation for their part in this enterprise. Apart from the defective diagram already mentioned, the only printing defects noted were a few slight misalignments of subscripts and superscripts. The large pages are well set out and the mathematical printing is excellent.

B. C. B.

Introduction to the Theory of Statistics. By G. Udny Yule and M. G. Kendall. Fourteenth edition. Pp. xxiv, 701. 34s. 1950. (Griffin)

The first edition of this book appeared in 1911 as the work of Mr. Yule. Since that time, in its successive editions, it has always held an honoured place among books on elementary statistics. After its first twenty-five years of useful life a major revision was called for, and Professor Kendall became co-author of the eleventh edition of 1937. Only minor changes were made in the editions of 1939 and 1944, but the new edition reflects some important developments of statistics that have taken place in recent years.

The book is too well known to need a detailed review of all its contents, but those who are indebted to earlier editions will be interested to know what is new. First there is a loss to record ; the extensive bibliography with its 680 references has been omitted. This is regrettable, but, as Professor Kendall says in his preface, a more extensive bibliography is now available in the second volume of his *Advanced Theory of Statistics*. Though the first four chapters of earlier editions, on the Theory of Attributes, have been condensed into two, the loss here is not serious. These losses, however, are more than offset by the gains, the most important of which are new chapters on Index Numbers, Time Series, Some Problems of Practical Sampling, and an expansion of the earlier treatment of Analysis of Variance. The chapter on Index Numbers is sufficient to satisfy all but the economic statistical specialist. Price-relatives, weighted and quantum index numbers are discussed with topical illustrative examples, and the principles of time-reversal, factor-reversal and circular tests are explained. The first of two chapters on Time Series deals with weighted moving averages ; their application to the analysis of trend, short-term systematic movement and random fluctuations ; and their possibilities of generating spurious oscillations in the random com-

ponents of time series. In the second chapter are considered time series from which the trend, if present, has been eliminated; for the analysis of the remaining fluctuations the method of serial correlation and the use of auto-regressive series are briefly described. In an entertaining chapter some Problems of Practical Sampling are discussed, among which are the effect of changing the size of the sampling unit, elaborations of sampling techniques (with a mention of Sequential Analysis), the cumulative effects of bias, and various phenomena such as the "vanity" and "sympathy" effects that occur in personal interview enquiries. The important chapter describing practical difficulties in Correlation and Regression has been entirely re-written.

In addition to these major changes there are many lesser amendments, mostly additions of new material. In view of Professor Kendall's interest in rank correlation, it is not surprising to find that a description of his coefficient τ is included. It is disappointing, however, to find that on this subject his powers of exposition fail; the description of the computing of the score S in Section 11.18 is too condensed. Other topics which are expanded, or are mentioned for the first time, are Gini's coefficient of mean difference (Ch. 5), the problem of finding a unique straight line of "best fit" when both variables are subject to error (Ch. 15), the significance of the regression coefficient (Ch. 21), and analysis of variance (Ch. 22).

The original purpose of the work was to give a systematic introduction to statistical methods to those with limited mathematical knowledge, and this has been remembered in the writing of the new sections. The resolve to admit no methods requiring the use of differentiation or integration (though results are quoted) may have been necessary forty years ago, but it is becoming a little irksome today. It is difficult to imagine modern readers able to profit from this book who are entirely ignorant of some elementary calculus. Such readers are, for example, referred to tables of the incomplete beta-function (p. 494) and assumed to have had some experience of harmonic analysis (p. 641). Some weakening of the resolve is apparent in the description of frequency curves (p. 416). Yet as a readable and comprehensive introduction to statistics it has no equal, particularly for those who must perforce teach themselves. Its examples and exercises are interesting and well chosen to illustrate the techniques described; the numerical answers are accompanied by helpful hints and comments. All the necessary statistical tables are included.

The book has deservedly become a standard work, and has already been translated into Spanish and Portuguese. The new edition ensures it another span of useful life.

B. C. B.

Lehrbuch der Funktionentheorie. By H. Hornich. Pp. vii, 216. 33s. 6d. geb. 37s. 1950. (Springer, Vienna)

This nicely written and nicely printed book covers a surprising amount of ground without making the argument seem unduly compressed.

In the first half of the book the elementary theory of uniform functions is developed in six chapters along fairly standard lines up to Laurent's series and the study of isolated singularities. It begins with the definition of complex numbers, their elementary arithmetic and geometrical representation, and fundamental ideas of convergence. Chapter II introduces the concept of differentiability, the Cauchy-Riemann equations, elementary ideas of conformal representation, and contains a study of the bilinear function

$$(az+b)/(cz+d).$$

Elementary converses of the fact that differentiability leads to conformal

representation are established. In the third chapter power series and the regularity of their sums are studied. After a chapter on integration along rectifiable curves we come to Cauchy's theorem. Jordan's theorem is quoted without proof, and many topological difficulties are avoided by proving Cauchy's result only for the integral of a function along a closed rectifiable curve lying in the interior of a simply-connected domain in which the function is regular. The author seems to have overlooked this when he shows how to express functions in terms of boundary values; there should have been a hint that simple approximations of the type described earlier enable integrals along a circle inside a domain to be replaced by integrals along the *circular* boundary when the function concerned is continuous there.

Among the applications of Cauchy's theorem there is a section on inverse functions in which estimates are found for the radius of the circle $|z| < d$ in which $w = a_1 z + a_2 z^2 + \ldots$ $(a_1 \neq 0)$ is schlicht, and of the radius of the circle $|w| < \delta$ in which the inverse function is regular.

It is difficult to know where to stop after embarking on the applications of the theorem of residues. The author wisely restricts himself to the evaluation of one definite integral and the formula for the difference between the number of zeros and poles of a function.

The second half of the book consists of four chapters designed to stimulate interest in more advanced topics of complex variable theory. The first two are introductory; they include the Weierstrass and Mittag-Leffler constructions of integral and meromorphic functions and a nice introduction to analytic continuation and to Riemann surfaces.

The next chapter discusses a variety of topics; we find here some results on conformal representation, including the existence theorem on the representability on the unit circle of a simply-connected domain whose boundary contains more than one point, Picard's theorem proved by the Bloch-Landau method, and some properties of Beta and Gamma functions.

Perhaps the most interesting chapter is the last. A discussion of algebraic functions and the associated Riemann surfaces leads to the study of integrals of algebraic functions. A special case is shown to lead to doubly-periodic functions. Finally, the author develops the elements of the theory of the Weierstrassian $\wp$, ζ and σ-functions, and shows their relationship to functions on a Riemann surface of genus 1. Unfortunately the proof of the periodicity of $\wp$ is defective; in trying to avoid the half-periods the author has been impaled on the poles by basing his argument on differences of values *at* the poles.

Examples are provided at the end of each chapter, and in the text the general theory is often illustrated by applications to particular functions.

The book provides a pleasant compact course in complex variable theory, and should stimulate its readers to embark on further studies. R. C.

Integraltafeln. Sammlung unbestimmter Integrale elementarer Funktionen. By W. Meyer zur Capellen. Pp. viii, 292. Dm. 36. 1950. (Springer, Berlin)

The author of this collection of indefinite integrals decided that there was a place for a comprehensive list in connection with his own interest, methods of instrumental integration, as well as for general use. He suggests its use as a source of examples, with answers.

After a short description of methods of integration, the table is arranged in four main sections, according to the integrand; these are for algebraic integrands, those involving the elementary transcendental functions, products of these two, and fourthly, integrands of the forms $g(x) \log x$ and $e^x g(x)$. Four pages of introduction and a comprehensive list of substitutions are given

in the first section before the integrands which lead to elliptic integrals. The other non-elementary functions needed are defined, with references to standard works, in a final section which also has a collection of information about the elementary functions.

The use of such a table depends on its comprehensiveness, and here the numerous special cases given with each general formula seem to cover well its various appearances. The connection between related integrands may, however, be sometimes obscured by the form of decimal references used; to refer to an integral as "Section 2.3.2.4., No. 4.2.3.1.2." seems clumsy, but may be necessary among some 3,000. The layout of the book is pleasing, and a verification of a random selection of the results suggests that the list of errata is fairly complete. Surely, however, No. 4.4.1.1 on p. 8 should have $(1 \pm x)^3$ in the integral, and the integrand in the third column of No. 2.0.2 on p. 13 should contain $(a+bx)^{2/n}$. R. B. H.

Grundzüge der Galois'schen Theorie. By N. TSCHEBOTARÖW. Translated and edited by H. SCHWERDTFEGER. Pp. xvi, 432. 17.50 fl. 1950. (P. Noordhoff N.V., Groningen-Djakarta)

This comprehensive exposition of one of the most beautiful theories of classical mathematics is written in a classical style, where emphasis is laid on concrete problems and their solutions, and where the abstract structure of the conceptual edifice is amply clothed with a wealth of lively detail. The reader is guided gradually and without haste, and though the journey may be long, it is full of interest and presents many a striking view across the wide field of algebra. The author, who speaks with so much authority on this subject, explains in the Preface that he did not write this work in the style and spirit of the abstract school, which was founded by Emmy Noether and others and which has dominated algebraical publications in the last two decades. For, although abstract methods have been used with outstanding success in modern research, it may be questioned whether they are equally appropriate for the teaching of algebra. Students will gain a deeper understanding if they first become familiar with the original development of the main ideas, as presented in this book, and then turn to the more abstract treatment of the modern school.

There is no doubt that this book by Tschebotaröw and Schwerdtfeger provides an excellent introduction to Galois Theory, and indeed to a large part of algebra. It can be read by students who have only a modest knowledge of algebra, since the treatment is as elementary as the subject permits. Great care is taken to state the theorems clearly and to bring out their significance. Formal arguments are often preceded or followed by explanatory remarks. Frequent references will be found very helpful, as will the numerous illustrative examples and problems (unfortunately without answers). In short, the work has all the features that one expects of a really successful textbook. But it is not only to the beginner that this volume will appeal. Originality in the treatment of well-known results and inclusion of some otherwise rather inaccessible subject-matter, are sure to make Tschebotaröw's book interesting and valuable also to the more experienced student of algebra. Only a few points can be mentioned here: following modern usage, the Galois group is defined as the group of automorphisms of the root field of a separable equation, but a full account is also given of the older treatment where the group appears as a permutation group leaving all "rational" relations between the roots invariant. The latter view underlies Mertens' construction of the group by means of the *fundamental modules* of a polynomial, which provide a special kind of basis for all relations between the roots. There is also a noteworthy account, supplied by the translator, of A.

Loewy's generalisation of Galois Theory for non-normal extensions and the consequent generalisation of group theory (*Mischgruppen*).

A brief summary of the contents will indicate the scope of the book.

Chapter I. *Theory of groups*, as far as it is required for the development and the application of Galois Theory.

Chapter II. *Fundamental algebraical ideas*, including fields, symmetric functions, algebraic extensions.

Chapter III. *The Galois Group*. Full treatment of the two aspects referred to above.

Chapter IV. *Soluble equations*. This deals in great detail with the binomial equation of prime and composite degree, of the cyclotomic equation of soluble real fields and of the soluble quintic.

Chapter V. *Equations with prescribed group*. An especially interesting chapter containing an account of Galois Theory over a field of characteristic p, M. Bauer's construction of polynomials whose Galois group is the symmetric group, and an outline of the methods of Hilbert and E. Noether.

There is a useful appendix in which the simplest facts of elementary number theory are proved.

The reviewer feels that this book should be warmly recommended to all students of algebra, and he welcomes the appearance of a work which, differing as it does in style and tenor from most modern publications, is a particularly valuable addition to mathematical literature. W. LEDERMANN.

The Skeleton Key of Mathematics. By D. E. LITTLEWOOD. Pp. 138. 7s. 6d. 1949. (Hutchinson's University Library)

This book is the eighteenth in a series that deals with many subjects, including mathematics. The present work is one on algebra, which sets out to describe the ideas that lie behind various algebraic theories of the present day in a manner that may interest and enlighten the lay reader, be he a non-mathematician or else a mathematician whose main interests are in other fields. The author starts with a brief essay on mathematical abstraction, and continues by developing systematically and from first principles the ideas of number, congruences, polynomials, algebraic integers, groups, the Galois theory, followed by a more detailed exposition of the technical tools of algebra ; namely, analytical geometry, matrices and determinants, invariants and tensors, hypercomplex number and group algebra, leading to an account of group representation as developed by I. Schur and A. Young, with further extension of the author's own work into applications to invariant and quantum theory.

To cover such a programme within the limits of a short book might seem impossible : yet such is the case. It is achieved by drastic condensation, of a kind that demands hard thinking on the part of the reader. Each topic is introduced at a pleasantly elementary level that makes a good preparation for the reader, though one doubts whether the non-mathematician will make much headway here! Then follows a leap to the formal mathematical statement of the topic, which is not always easy to follow : one feels that condensation has perhaps gone too far. Next follows a passage exemplifying the idiom of the particular technique belonging to the topic : this is enlightening and interesting, owing both to the variety of subjects that come under review and to the care with which the examples have been kept as simple as possible while remaining appropriate.

By unlocking some of the intricacies of nuclear physics with his master

key, which is abstract algebra, the author has justified the choice of title for his book. But has he not whittled down the wards of his key a thought too much by omitting all reference to Cayley, Gordan and Hilbert?

H. W. TURNBULL.

Einführung in die Differentialgeometrie. By W. BLASCHKE. Pp. vii, 146. geh. DM. 16, geb. DM. 18.60. 1950. (Springer, Berlin)

This latest addition to the " Grundlehren " series is a very concise account of the differential geometry of curves and surfaces in which constant use is made of the calculus of differential forms developed by Cartan. There is thus very little overlapping in treatment, though much in content, with the author's well-known earlier work in the same series. The whole treatment is remarkably elegant, and the numerous examples give a very rich and varied set of applications of the theory. The book forms an interesting alternative to the more conventional methods of developing the subject. J. A. T.

Lectures on Classical Differential Geometry. By D. J. STRUIK. Pp. viii, 221. $6. 1950. (Addison-Wesley Press, Cambridge, Mass.)

This is an elementary textbook covering much the same ground as the well-known work by Weatherburn, and uses the vector calculus in much the same way. The most striking feature of the book is the way in which the author emphasises that the subject-matter is geometry and not vector algebra, and the explanations which accompany the exposition keep the geometrical point of view well to the front. The text is enriched by a splendid series of plates and diagrams, which not only enhance the appearance of the volume but make the visual aspects of the subject unusually clear. The exposition is lucid and accurate, and the book would be eminently suitable for an undergraduate course in our own universities. The reviewer has noticed only one slip: the statement on p. 55 that the vanishing of two of the determinants from the 2×3 Jacobian matrix implies the vanishing of the third, admits a trivial exception. The printing and layout of the book are excellent.

The five chapters deal respectively with curves, elementary theory of surfaces, the fundamental equations (Gauss theory), geometry on a surface (geodesics), and, finally, some special surfaces (ruled surfaces, minimal surfaces, mappings). There are numerous exercises throughout the book, with answers and hints for solution at the end. This book can be warmly welcomed. J. A. T.

Topology of Manifolds. By R. L. WILDER. Pp. x, 402. $7. 1949. American Mathematical Society Colloquium Publications, 32. (American Mathematical Society, New York)

A manifold, in almost any context, is a space which is locally like n-dimensional Euclidean space. Difficulties arise, however, when one asks exactly how close the resemblance has to be. In its classical use the term requires that each point have a neighbourhood homeomorphic to Euclidean space; and in applications this is indeed what one usually knows, or would like to know. But as a definition it is inconvenient, because it is very difficult to tell whether a space satisfies it. One may, for instance, consider the problem of proving that a point set in Euclidean space is a manifold, given appropriate conditions on its residual set. To do this using the classical definition one has to construct, as it were from nowhere, a set of coordinate systems on the point set, obviously a formidable undertaking. Ideally, the way out of this difficulty is to find an equivalent definition in terms of properties which are easier to test. This is the problem of characterising manifolds; but except in dimensions less than three it has proved altogether too difficult. Accord-

ingly, the theory of manifolds is now usually based on a wider definition. In a sense this merely shelves the main problem, but it has the great advantage of making a much more complete theory possible. On the one hand, the results of the classical theory, such as the duality theorems, are preserved, and on the other, it becomes possible to handle such questions as the recognition of submanifolds.

The work under review is an account of the theory of manifolds in the most general sense, in which all that is assumed is, roughly speaking, that the spaces under consideration are locally like Euclidean space in their algebraic topology, that is, as regards homology and cohomology theory. As befits a Colloquium Publication of the American Mathematical Society, the treatment is detailed, comprehensive, and up to date ; and the part played by Professor Wilder in the development of the theory is sufficient guarantee that it is authoritative. It would, of course, be too much to expect a book like this, in addition to its other virtues, to be easily readable. The large number of related but inequivalent concepts, and a formal apparatus which includes incidentally most of algebraic topology, preclude that. But, except perhaps in the elementary first chapter, the author has minimised these difficulties to a surprising extent, interspersing in the formal work paragraphs of comment to illuminate the direction that work is taking. The standards of book production reached by this series are too well known to need comment, and it is perhaps churlish to remark that it is not easy to relate figure T_1 on p. 198 with its description in the text. To sum up, this is a book for specialists ; to them it will be indispensable ; no one else is likely to make much of it. G. H.

Differential Algebra. By J. F. Ritt. Pp. viii, 184. $4.40. 1950. American Mathematical Society Colloquium Publications, 33. (American Mathematical Society, New York)

The subject-matter of this book is the abstract algebraic treatment of differential equations, with which the name of Professor Ritt has always been closely associated. The treatment is based on the concept of a differential field, that is, a field in which an operation of differentiation is defined. A concrete example, used as an illustration throughout the book, is the field of all meromorphic functions defined in some domain of the complex plane.

The book begins with a treatment of differential polynomials over F, that is, polynomials in a set of indeterminates $x_1, \ldots, x_n$ together with the derivatives of these indeterminates. The set of all such differential polynomials forms a ring in which a differentiation is defined. The ideal theory of this ring is then developed, attention being restricted to ideals closed under the operation of differentiation.

This ideal theory is then applied to the manifold of a set of differential polynomials, this manifold being the set of all solutions of the set of differential equations obtained by equating the differential polynomials of the set to zero, all solutions in any differential field containing F being considered. As in algebraic geometry, the ideal theory already mentioned leads to a decomposition theory of such manifolds, an interesting application being a precise formulation of the definitions of the general and singular solutions of a differential equation.

It is with the development of the ideas mentioned above that the book as a whole is concerned. Particular emphasis is placed on constructive methods, one example being the development of an elimination theory for sets of differential equations. The book also contains chapters on the theory of ordinary algebraic equations, developed by the methods used for differential equations, and on analytic considerations, and concludes with a chapter on partial differential polynomials.

The abstract nature of the subject-matter makes the book, of necessity, difficult to read for anyone not acquainted with the spirit of modern abstract algebra. However, the author has attempted, successfully, in the reviewer's opinion, to avoid the requirement of the knowledge of specific results in that discipline. Judicious use of concrete examples throughout the book helps the reader to understand the motivation behind the results contained in the book.

D. REES.

Elemente der Analytischen Geometrie. 1 Teil. Studia Mathematica, Band II. Pp. 232. DM. 14 ; geb. DM. 16.50. 1948.

Elemente der Analytischen Geometrie. 2 Teil. Studia Mathematica, Band V. Pp. 156. DM. 8.80 ; geb. DM. 10.50. 1949. (Vandenhoeck & Ruprecht, Göttingen)

Vandenhoeck & Ruprecht of Göttingen are new names to most readers of mathematical books. In spite of the difficulties of the times, they have produced a series (Studia Mathematica) of most attractive textbooks, and we are promised several more, some being of a more advanced character than these, which will be awaited with the very greatest interest. The books are pleasantly bound and nicely printed : it is to be hoped that the publishers will have the success their enterprise deserves, and that they will long continue to earn the gratitude and good wishes of the mathematical community. Perhaps in view of the difficulties they have had to contend with, it is rather ungracious to insert minor complaints of a lack of uniformity in the production of these two volumes ; they are not of the same size, and the table of contents is at the end of Vol. I and at the beginning of Vol. II.

The first thing that takes the eye in these books is the annotation, which reveals many fascinating items of historical interest : for these the author generously apportions the credit to Herr J. E. Hoffman, from whose erudition it is to be hoped we may profit further in future works. On the very first page of Vol. II there is a footnote on the history of permutations. Here it is remarked that the more important properties of these are already to be found in a Hebrew work of 1321 due to a certain Rabbi Levi ben Gerson, a scholar at the papal court of Avignon, who was also, according to a later footnote, acquainted with the concept of a group. Equally remarkable is the reference in Vol. I to the seventeenth-century Japanese mathematician, Kowa Seki, who, among other achievements, had discovered the method of solving simultaneous linear equations at about the same time as his European colleagues. Each of these volumes contains not only an adequate index, but also short biographies of all the mathematicians mentioned in the text or the footnotes.

Vol. I treats, in six chapters, of plane and solid geometry, including the use of vector methods and homogeneous co-ordinates, contains a chapter introducing the fundamental notions of projective geometry of one dimension, and deals with circles, spheres, conics and quadrics. Vol. II contains three chapters only, which deal with determinants and matrices, systems of linear equations, linear transformations (including invariant factors) and an introduction to the place of group-theoretical ideas in Geometry. A third volume is forthcoming on Projective Geometry, and with the groundwork laid here it is to be expected that this will do an outstandingly good job. The presentation is clear and intelligible throughout. The underlying algebra is neatly and accurately presented and put to very good use ; the argument is nowhere marred by inexact or inelegant methods, and unlike many works which deal fairly with the algebraic side of the subject, this one is always animated by a real geometrical feeling.

This work is to be heartily recommended to teachers and even more strongly to students, especially first and second year undergraduates and bright sixth form pupils. It is not designed as a text for English examinations, though it must make an admirable course-text for the author's students. But it is, in the reviewer's opinion, a model of what we may fairly expect from a foreign-language textbook. It is easy and interesting to read: it deals with matter which will mostly be familiar to the student, and enables him to get valuable practice in reading mathematical German without placing too great a strain on either his linguistic or mathematical powers. He will find, especially in reading the first volume, that a great deal of familiar matter has been presented in a possibly unfamiliar way which he is bound to find interesting and stimulating. Even though he be quite at home with the subject-matter, he will find his grasp strengthened and his horizon widened. What more can any teacher ask for his pupils? D. B. S.

Colloque de Géométrie Algébrique, Liège, 1949. Pp. 195. 200 fr. belges. 1950. (Thone, Liège)

This book consists of a series of papers given at the Colloquium held at Liège on 19th–21st December, 1949, under the auspices of the Centre belge de Recherches mathematiques. This was the first international colloquium organised by the Centre, and was devoted to Algebraic Geometry.

The work opens with a long paper by Severi on Algebraic Geometry in Italy, its rigour, its methods and its problems. Various questions concerning the Cayley form associated with an algebraic variety are investigated; and the theory of elimination is discussed from the geometric point of view. The importance of abstract algebra and topology and the theory of algebraic functions and their integrals for algebraic geometry is emphasised, and the interaction of these various subjects is illustrated by a discussion of what Severi considers to be the most important problems. These include problems on (1) the theory of the base, (2) series and systems of equivalence, (3) the periods of integrals of the first kind on an algebraic variety, (4) surfaces possessing a discontinuous group of birational self-transformations, (5) the irregularity of, and the Riemann-Roch theorem on, a variety, (6) quasi-Abelian functions and varieties. A valuable bibliography (79 references) is given.

The next paper is by Dubreil and Madame Dubreil, and is an exposition of the various types of rings which arise in algebraic geometry. Then comes a paper by van der Waerden on chains of varieties on an abstract variety (in the sense of André Weil); and a paper by Samuel on the multiplicities of singular components of intersection.

The applications of algebraic geometry in other branches of mathematics are dealt with in papers by Segre, Garnier, Châtelet and Bureau. The paper by Châtelet is concerned with an application of the ideas of Galois theory to algebraic geometry, and that by Bureau discusses some geometric problems suggested by the theory of partial differential equations which are totally hyperbolic. Segre's paper concerns *arithmetic* problems in algebraic geometry. It commences with a problem in diophantine analysis: to find all integral solutions of the equations

$$x_1^2+y_1^2=x_2^2+y_2^2=x_3^2+y_3^2,$$
$$x_1^3+y_1^3=x_2^3+y_2^3=x_3^3+y_3^3.$$

This problem is not easy unless one adopts a geometric approach, and Segre has shown how powerful this method can be. The other topics dealt with by Segre are the theory of quadrics in an arbitrary commutative field, and the generalised Severi-Brauer varieties. The paper by Garnier deals with an

application of algebraic geometry to a problem in the theory of functions. The problem of determining the condition on a polynomial $f(x, y)$, in order that the solution of the differential equation $f(y', y)=0$, where $y'=dy/dx$, shall be a *single-valued* function, is well known. The condition is simply that the genus p of the curve $f(x, y)=0$ must satisfy $p \leqslant 1$. Garnier's generalisation of this problem is: to determine the condition on an algebraic surface, $F(x_1, x_2, x_3)=0$, and the rational functions R_{ijk} of a point of F, such that the general integral of the system of equations

$$\frac{\partial x_i}{\partial u_j}=w_{ij}, \quad \frac{\partial w_{ij}}{\partial x_k}=w_{ij}R_{ijk}(x_1, x_2, x_3) \quad (i, j, k=1, 2)$$

shall be single valued. It is found that the conditions can be expressed in terms of the genera p_g and p_a of F. Garnier observes that we must have $p_g \leqslant 1$. He then examines separately the cases where F is (i) irregular, (ii) regular. In (i) it is found that if $p_g=1$ we must have $p_a=-1$.

Finally there are papers, by Libois and by Godeaux, giving an account of recent research in Belgium. Libois discusses the use of various notions of abstract algebra in geometry, and the paper by Godeaux deals with applications of the theory of cyclic involutions on an algebraic surface. The surfaces dealt with have either $p_a=P_4=1$ or $p_a=p_g=0$, $P_2=1$. A construction is given for various irregular surfaces.

From the above survey it will be evident that in this report algebraic geometry is discussed in its widest aspects, and the intimate connection of the subject with other important branches of mathematics is emphasised. This comes as a tonic to one filled with a sense of the growing diversity in present-day mathematical research. Algebraic geometers will be grateful to Professor Godeaux for his organisation of the colloquium and for the publication of this excellent report. L. S. G.

Fourier Transforms. By S. Bochner and K. Chandrasekharan. Pp. 219. \$3.50 (28s.). 1949. Annals of Mathematics Studies, No. 19. (Princeton University Press; Geoffrey Cumberlege, Oxford University Press)

A number of topics covering the parts of the real variable theory of Fourier transforms important for applications and the related General Transforms are dealt with in this book. It begins with the theory of Fourier transforms of functions on $L(-\infty, \infty)$, treating their convergence and summability theory, with some relatively elementary uniqueness theorems. There is a thorough discussion of summability by a very general class of kernels, and applications of these results to give theorems about ordinary convergence. The second chapter deals with functions of several variables, containing a number of contributions which the authors have made to the subject, particularly the theory of summability by means of radial kernels. Chapter III discusses L_p spaces, giving some account of the general theory of these spaces and of Banach spaces in general, but little of the theory of Fourier transforms in these spaces. Chapter IV develops the theory for L_2 spaces, both in one and in many variables, very thoroughly. Chapter V, after a preliminary account of general unitary transforms on $L^2(0, \infty)$ induced by integral transforms, deals with Watson transforms, proving a number of interesting results not hitherto available in the books on the subject. The last chapter deals with Tauberian theorems, whose study is begun in the first: the treatment is, fundamentally, that of Wiener, but there are many new ideas, and the proofs and results differ from those of Wiener and his collaborators.

The attitude of the book is modern, making consistent use of the theory of linear spaces. This gives particular liveliness to the treatment of the Watson transforms. The section on multiple Fourier transforms fills a very definite

gap in the literature, made the more glaring by the many applications of these transforms based on no rigorous foundation.

The application of Fourier transforms to the solution of differential equations is little discussed, presumably because it is so fully treated elsewhere, and also because the purely real-variable approach of the book would make strong restrictions on the size of solutions at infinity necessary : this is illustrated by the two cases discussed, the heat conduction equation and the potential equation which arise in connection with Gaussian and Abelian summability respectively.

As a whole the book is very readable, and could be studied profitably by anyone with a knowledge of the Lebesgue integral. The one exception is the last chapter, where the analysis is rather heavy. The production of the book, by photography from typescript, is well and clearly done. There seem to be very few errors, mainly typographical. One might mention that the name of Plancherel is fairly consistently misspelt, and that in theorem 79′ on p. 158 the number (2.9) in the third line should be (2.10), and in the last line the numbers should read (2.6), (2.7) and (2.9) respectively. The remarks in the 2nd and 3rd paragraphs of p. 215 are incorrect. J. L. B. C.

Functional Operators. By John von Neumann.

Volume I : Measures and Integrals. Pp. i, 261. 22s. 6d.

Volume II : The Geometry of Orthogonal Spaces. Pp. 107. 25s. 1950. (Princeton University Press. London : Geoffrey Cumberlege)

These two volumes contain lectures given by the author at the Institute for Advanced Study in 1933–5. They have exerted considerable influence on students of Measure Theory and Operator Theory in the U.S.A., and are now reproduced, with some corrections, as Nos. 21 and 22 of " Annals of Mathematics Studies ".

The two volumes can be read independently of one another : the first is not restricted by any particular needs of operator theory, but contains a complete account of the theory of integrals of summable functions. The second makes little use of matter in the first.

The first nine chapters of Volume I treat the theory of Lebesgue measure and integration in n-dimensional spaces by more or less classical methods, including the Vitali covering theorem and the theory of the differentiation of the integral. With the section on Fubini's theorem in the last chapter these chapters would cover pretty well all an analyst needs to know about Lebesgue integration. Chapters X and XI each occupy about one third of the volume, and deal with the generalisation of the theory to abstract spaces. Chapter X gives the theory of extension of measures for rings and other algebraic systems of sets, and also the theory of measures on product spaces : this theory is used in an ingenious fashion to give a definition of the ordinary Lebesgue integral and of Lebesgue-Stieltjes integrals free from the normal topological notions. Chapter XI deals with the general integral, discussing in great detail the derivative of one set function with respect to another. The discussion almost reverses the normal order, and though advantages are claimed for it the treatment seems rather heavy.

The second volume is devoted to the geometrical theory of orthogonal vector spaces, those in which a scalar product is defined. These spaces are defined axiomatically, and examples of spaces obeying various combinations of the axioms are given. This is followed by definitions of the main types of operators, and by what, if it is not stretching that abused adjective, we may call the elementary theory of these operators : their algebra and geometry : the analytic resolution theory is not touched. Most of the contents of Volume II are due to von Neumann ; but although it contains much not in his pub-

lished papers, it is far from exhausting or even giving the most important parts of his great contributions to the subject. There are, in fact, references to chapters yet to come.

All definitions and proofs are given in great detail and the treatment is very thorough and accurate. It is likely that Honours mathematics would suffice for reading all save a few passages. However, the book has some of the defects of lecture notes : it is rather long, owing partly to repetitions of arguments, and lacks the remarks intended to orientate the reader which would be given verbally in a lecture. An index of definitions and of notations would also make it more readable ; and references, if only by giving the names under which some of the classical theorems are known, would be very useful. The typography is good, and there are few typist's errors. J. L. B. C.

Measure Theory. By Paul R. Halmos. Pp. vii, 304. \$5.90 (45s.). (Macmillan, London ; D. van Nostrand, New York)

This is the second in the University Series in Higher Mathematics. It gives a comprehensive account of those aspects of the theory of measure and integration which are important in general measure spaces and in topological spaces and groups. After a discussion of algebraic systems of sets, measures and the extensions of measures on such systems are dealt with, followed by the theory of measurable functions and of integration. The main treatment is for the abstract case : but the Lebesgue integral on the real line is discussed in detail *pari passu* with the general discussion. There follows a chapter on general set functions, in which the decomposition of a completely additive set function and expression of an absolutely continuous set function as an integral is treated. The next chapter deals with product spaces, both finite and infinite. Chapter VIII, on Transformations and Functions, deals with measure rings and their transformations, and has an account of the general theory of Boolean rings. The following chapter, on Probability, begins with an account of the considerations which lead to the identification of the mathematical theory of probability with the theory of measures on Boolean rings, and then gives an excellent account of some of the major results in the mathematical theory of probability, including the central limit theorem.

The last three chapters are devoted to very recent developments, the theory of measures on locally compact spaces, and the theory of the Haar measure on locally compact groups. The last chapter discusses how the knowledge of a Haar measure determines the topology of such a group. These sections are certainly the most readable account of this subject in print.

The exposition throughout is masterly. Proofs are clear, precise and elegant. Care has been taken to avoid the indigestion which accompanies the reading of abstract theories, by discussions in each section aimed at giving the reader an intuitive grasp of the subject and an idea where he is going, and by well-chosen sets of examples. These last should not be overlooked by the reader : they contain not only helpful illustrative matter on the main text, but also accounts of further developments of the subject.

The first seven chapters require from the reader a knowledge of Honours mathematics ; the last three need in addition a knowledge of topology and group theory. The book deserves to become a standard work on its subject. For the analyst interested in real variable theory it is complete save for a treatment of certain subjects which have been omitted, evidently because they do not generalise to abstract spaces : principally, the theory of differentiation of the Lebesgue integral, and also, of course, the Denjoy and such integrals.

The text is well printed and accurate. The book ends with a bibliography, exclusively of the more recent works, and an index. J. L. B. C.

Analyse Harmonique. Pp. 132. 13s. 1949. Colloques Internationaux du Centre National de la Recherche Scientifique, 15. (C.N.R.S. and Gauthier-Villars)

This contains papers given to a Colloquium held in Nancy in 1947, which was attended by some of the major foreign as well as French authorities on Harmonic Analysis. The term is understood in a wide sense, and the papers form useful introductions to recent work on the theory of Fourier series and integrals and their generalisations from the point of view of real and complex function theory ; three papers, two by Wiener and Paul Levy, treat applications to statistical theory and random variables. The first paper is a brief account by Laurent Schwartz of his generalised functions. The book can be strongly recommended to anyone wanting a short survey of recent research in these subjects. France is to be congratulated, both for its initiative in arranging the colloquium and for the excellent work contributed by the French mathematicians. J. L. B. C.

Les fondements logiques des mathématiques. By E. W. Beth. Pp. 222. 1400 fr. 1950. Collection de Logique Mathématique, Serie A. (Gauthier-Villars)

In the hundred years which have elapsed since the publication of Boole's *Formal Logic*, the scientific study of the foundations of mathematics has progressed so rapidly that even a superficial survey of the whole range of recent achievements would fill a substantial volume. By a strict exclusion of detail, however, and a style that is as crisp as it is efficient, Professor Beth has succeeded in compressing into less than two hundred pages an account of a wide range of topics that is as readable as it is authoritative.

After a short historical introduction the Dedekind and Peano definitions of natural numbers are described and contrasted, and there is a very brief reference to the construction of integers, rationals and real numbers. Beth follows van der Waerden in taking an integer to be a *class* of equivalent pairs of natural numbers, but there seems to be no advantage in introducing both *equivalence* and *equality* for integers. A simpler theory is obtained by taking the individual ordered pairs themselves to be the integers, two pairs (a, b), (c, d) being equal or unequal according as $a+d$ does or does not equal $b+c$.

The section on numbers ends with an important observation on a result obtained in 1939 by the Polish logician Tarski on the existence of a proposition in the realm of real numbers which is not reducible to propositions about natural numbers.

Symbolic logic and the theory of proof form the major part of the next section, which is followed by an account of the logistic systems of Frege and Bertrand Russell. The account of Gödel's theorem (p. 85) is unfortunately vitiated by the lack of detail. Beth himself says (p. 87) that his simplified presentation does scant justice to Gödel, and in fact it is in the detailed construction of a considerable edifice that the great merit of Gödel's work lies.

The remaining sections of the book deal with the theory of sets, intuitionism and the logical paradoxes. The paradoxes are enumerated at some length and separated into various categories, but this section also suffers in part from a lack of detail, for instance in regard to the theory of types and Russell's axiom of reducibility.

As one would expect from a scholar of the Dutch school, Brouwer's *intuitionism* is sympathetically handled, and though there is no more than a passing reference to intuitionist analysis, the spirit of that work is successfully conveyed in the proof of the non-existence of the complement in intuitionist set theory. There is no mention of finitist notions other than those of the intuitionists.

In view of its modest size, it is no adverse criticism of the book to mention what appear to be the major omissions. Nothing at all is said about the theory of recursive functions, and considering the exceptional importance of this theory in present-day work on the foundations of mathematics, the omission is a serious one; for lack of it, no detailed exposition of Gödel's work, for instance, is possible. It may be, however, that the omission is a matter of policy and that a separate volume on recursive functions is contemplated in the same series. Logic-free formalisations of arithmetic is another (but related) topic which is conspicuous by its absence.

Despite the set of exercises with which the book concludes, this is not a textbook for a beginner who seeks to acquire a new technique, but rather, to borrow a phrase from current medical terminology, a " refresher course " for the general practitioner. The method is encyclopaedic rather than philosophic, richer in reference than in criticism. In a field which was once notorious for the controversies which it engendered, Beth's self-restraint and impartiality sound the bell of a new age which has rediscovered the virtue of tolerance, a virtue that is, alas, as often the symptom as the cause of the onset of sterility.

Symbolic logic appears to lend itself well to lithographic reproduction. The limitation to typewriter symbols is unfortunate in only one instance, the use of a capital V as a disjunction symbol in such a context as UVV (page 47, 2*b*, and page 151, 1.14). Misprints are few and of a minor kind, as the following list shows:

Errata.

p. 17, (*b*), 1.2. For " u " read " v ".
p. 29. Lacuna in definition of Q, read " $Q(e')=e$ ".
p. 43. For " $=d$ " read " $=_d$ ".
p. 47, (2*a*). For " U " read " $\bar{U}$ ".
(2*c*). For " $V \longleftrightarrow U$ " read " $V \rightarrow U$ ".
p. 80, 1.4. For (1*a*) read (2*a*).
p. 85, last line. For " G " read " g ".
p. 120, 1.2 up. For " n, n' " read " m, m' ".
last line. For " $<'$ " read " $<$ ".
p. 146, (4). For " $1/2.\surd 2$ " read " $(1/2)\surd 2$ ".
p. 148, last line. For " c " read " 3 ".

R. L. GOODSTEIN.

Cours de Mécanique Rationnelle (Tome iv, Problèmes et Exercices). By LOUIS ROY. Pp. xi, 276. 1200 fr. 1950. (Gauthier-Villars)

The first three volumes of this work appeared during the years 1944 and 1945, and were based on lectures delivered at the " Faculté des Sciences de Toulouse ", where the author is a professor. This present volume, which consists entirely of worked examples, is intended to supplement the second and third volumes, dealing respectively with *The Motion of a Particle* (culminating in *Systems of Particles*), and the problems of *Continuous Deformable Media.*

In all, there are fourteen chapters of problems split up by the four headings: " Kinematics ", " Single Particle ", " Systems of Particles ", and finally, " Continuous Deformable Media ". However, as may well be expected, seven of these chapters are devoted to the third of these headings, insomuch as the greatest variety of problems is to be found here. In fact, this " Troisième Partie " commences with a chapter on " Systems with Frictionless Constraints ", whilst later on we find chapters devoted to " The General Motion of Rigid Bodies ", " Impulsive Motions ", and " Newtonian Potential ".

The standard of the problems is such that the book would prove very useful to the student reading for an honours degree in mathematics, and most refreshing to those who have already concluded such a course. The examples are everywhere worked out with meticulous care and with an eye to detail, whilst in numerous cases alternative methods of solution are indicated.

The standard of the printing is excellent and the diagrams beautiful in their neatness and clarity. J. WILLIAMS.

Éléments de Physique Moderne Théorique. I. Mécanique Ondulatoire. By G. GUINIER. Pp. 159. 1949. (Bordas, Paris)

This is a straightforward textbook of wave-mechanics. It seems to the reviewer that it is well-adapted to give a sound working knowledge of the subject to students who are " encore un peu débutants ", in the charming phrase of L. de Broglie's preface.

An introductory chapter on " Waves and Corpuscles " gives an elementary account of some of the basic notions of quantum physics up to and including de Broglie's electron waves. The second chapter is on " Schrödinger's Wave Mechanics ", and presents fairly fully the mathematical treatment of its applications to one-body problems. There is only a brief sketch of the extension to more general systems. The third chapter on " Principles of Wave Mechanics " is a simple account of the correspondence principle, the uncertainty principle, the exclusion principle, electron spin, and immediately related topics.

The formulation of the fundamental equations is somewhat cavalier. Also one notes that, for instance, the boundary conditions for the solution of the wave equation, when first introduced (p. 53), are merely stated without physical reasons being given. However, there is probably a good case for letting the student see as soon as possible how the theory works before troubling him too much about its foundations. So the book could form a valuable basis for, or adjunct to, an introductory course of lectures on quantum theory.

This book, almost in its entirety, could have been written any time in the last twenty years or more. On the whole, this is a satisfactory thought. For it shows how lasting has been the value of the early work on wave mechanics, not only in its basic concepts, but also in the methods first used for its applications. W. H. McC.

Mechanics and Properties of Matter. By R. C. BROWN. Pp. ix, 276, ix. 10s. 6d. 1950. (Longmans)

This is the first volume of a Textbook of Physics which " is intended to be a straightforward presentation of the principles of Physics up to approximately the standard required by the Higher School Certificate and Intermediate examinations ". It will be welcomed by teachers of mathematics, for if their physics colleagues use this book there will be little conflict between the two treatments of mechanics. We shall readily agree that the " significance of the last line of a mathematical treatment of a physical phenomenon depends as much upon the physical principles incorporated in the first line as upon the elegance of the mathematics itself ". And we shall agree about the importance of a proper understanding of the physical principles or assumptions even when, as is so often the case, there is little elegance.

The first half of the book is devoted to the dynamics and statics of a particle and a rigid body, whilst in the second half there is the usual hydrostatics, a short chapter on fluid motion, and separate chapters on elasticity, viscosity, diffusion and osmosis, and finally surface tension. Most of the examples are drawn from examination papers; they are often few in number, but they are likely to be enough for a student of physics, though inadequate for one

taking applied mathematics as a subject in the Higher School Certificate examination.

The discussion of Newton's laws of motion is interesting, and makes clear the basic concepts and ideas involved. The author adopts the standpoint that the fundamental method of determining the mass of a body is by means of a (rather idealised) collision experiment. He later adds : " The question of the weights of the two bodies does not enter into the discussion ; indeed, the experiment would be easier to perform if, for the time being, the effect of gravity on the bodies (*i.e.* their weights) could be eliminated." It certainly would, but it is not very convincing. One suspects that students at this stage will not find the argument easy nor will they appreciate its necessity ; a discussion as fundamental as this might well be postponed until the student has had more experience of the subject and its methods.

The equations of motion of a rigid body are introduced in a novel manner, the centre of mass being defined as the point of a body such that if the line of action of an external force passes through this point no angular acceleration is produced by that force alone. The identity of the centre of mass and centre of gravity is established by appeal to experiment. The development is interesting and generally clear, but a statement such as " Forces and couples *are really abstract quantities* which we postulate as being the causes of acceleration " must tend to confuse the student.

Passing to statics, a rigid body is said to be in equilibrium when it has neither linear nor angular acceleration, and the conditions of equilibrium are thus derived from the equations of motion.

The hydrostatics is straightforward ; it does not include the calculation of centres of pressure. The treatment of the properties of matter topics is orthodox except for surface tension, the author believing in the " reality " of surface-tension forces.

Enough has been said to indicate that this volume has unusual features. There are few blemishes. One or two of the figures could have been improved by including *all* the forces, *e.g.* Figs. 85 and 87, and in Fig. 68 the frictional force might have been shown in its actual position. Centrifugal force is mentioned, but happily is not used to solve problems of circular motion. The equation of simple harmonic motion might, in several examples, have been better written with due regard to signs.

But these are all small matters ; this book contains the most attractive treatment of mechanics in a textbook of physics that the writer has had the pleasure to read. J. TOPPING.

Graphs. By R. J. GILLINGS. Second edition. Pp. 79. 5s. 1950. (Australasian Publishing Co., Sydney, N.S.W. ; Harrap)

The first edition of this book from Australia was printed in what appeared to be hand-written capitals ; it was reviewed in the *Mathematical Gazette* for December 1945. This new edition is in ordinary type, and the figures are drawn on paper ruled in squares of sides $\frac{1}{20}$th of an inch. The figures are very clear, but it is a pity that those from which readings have to be taken are not on double the scales used.

The major part of this edition is a reprint of the former edition, and does not need further comment than was made on that. The additions are :

(i) Graphs showing distance and velocity plotted against time, and distance plotted against velocity. There is one travel graph drawn, and only one for the student to draw for himself.

(ii) A section on cubic and other curves. In this the 42 cases are considered of the graphs of

$$y = ax^3 + bx^2 + cx + d + \frac{e}{x} + \frac{f}{x^2},$$

when at least three of the coefficients are zero. They are built up by considering the cases in which only one coefficient appears ; then, by combining these graphs, the cases in which two coefficients appear, and finally when three coefficients appear. This section gives figures for the various graphs when each coefficient is unity, and the accompanying letterpress consists mainly of statements of properties of the curves ; it would have been better if the student had been left to sketch the curves and find the properties for himself—the finding of the properties would have been more useful than a knowledge of the properties. A. W. S.

Storia delle matematiche dall' alba della civilta al secolo XIX. By Gino Loria. Second edition. Pp. xxxv, 975. 3800 lire. 1950. (Hoepli, Milan)

Loria's excellent *History of Mathematics* first appeared 1929 to 1933 in three squat volumes, each of from five to six hundred pages, published by Spen of Turin at 75 lire per volume (bound). The second edition is in a single volume of nearly a thousand large pages, and the price is 3800 lire (unbound). It is said to be revised and brought up to date, but a fairly careful collation with the first edition shows that for the most part it is a straight reprint. There are additions to the bibliographies, at the ends of the chapters, of books and articles published since the first edition ; an account of Neugebauer's work has been added to the section on Babylonian mathematics (pp. 8–10) ; a new footnote (p. 383) gives Ludolph van Ceulen's epitaph (which contains the value of π correctly to 35 places of decimals—but there is a misprint in Loria) ; there are small additions to the accounts of the Danish mathematician George Mohr (p. 550), of Leibniz (p. 588), and of Brill and Noether (p. 887) ; there has been a slight rearrangement of Chapter 35 ; the title of Chapter 10 has been changed from " At the Foot of the Himalayas " to " In the Land of Sanscrit " ; but that is about all. Even mis-spellings of the titles of English books—regrettably frequent in Italian printing—have been faithfully copied from the first edition—and new ones have crept in. But the book remains a great *tour de force*, interestingly written and containing a vast amount of information. And on the whole the new edition is pleasanter to use than the old.

The preface conjoins the three summaries of the old edition, but adds at the beginning a short but interesting discussion of the two types of histories of a particular science, panoramic accounts for the general reader, in which biographies of the giants of the subject must bulk largely, and those addressed to professionals. Loria proclaims that he unhesitatingly chooses the second, and so writes " as a mathematician for mathematicians ". F. P. W.

Plane Trigonometry. By G. Fuller. Pp. 189, xii + pp. 70 tables. 23s. 6d. 1950. (McGraw-Hill)

The author describes this book as a First Course for College Students. Although it is intended for University students, the standard is something between " School Certificate " and the first year of " Higher School ". The author assumes on the part of the student complete ignorance, not only of this subject, but also of the meaning of significant figures and ability to do logarithmic calculations. He also assumes that the student has a low standard of mathematical attainment, and considers it necessary to define such terms as identity in algebra. He decides that it is necessary to describe everythings in terms of simple language. Nearly every formula is restated, in words, even such a formula as

$$\Delta = a^2 \sin B \sin C / 2 \sin A,$$

for the area of a triangle. That this is of little value is obvious. The book is written in clear and simple language, but simplicity is overdone when we have: "The principle of interpolation assumes that a function of an angle is proportional to the angle." When the author has got halfway through the book, he has not progressed beyond the solution of the right-angled triangle. We then have, in a single chapter, compound angles and the product formulae. This, by contrast, is certainly too fast for the type of student for which the book is written. To offset this, the examples are all very easy. Indeed, they are easy throughout the book, and there is little to extend the better student. Although the book gives the formulae for changing a sum or difference into a product, there is no reference to the formulae for the converse process. The chapter on the solution of equations deals only with the simplest types and the formulae for the general solution are not given. The graphs of the circular functions come near the end of the book, an arrangement which would not meet with favour here, and there is no attempt to deal with the graphical solution of equations. The last chapter is a brief treatment of complex numbers, and the answers at the end are for the odd numbers of the exercises only (why?). The last 70 pages are occupied with tables of the circular functions and their logarithms.

It is clear that the American freshman enters the University with a standard considerably below that of his opposite number here. It seems that where a system of Comprehensive High Schools prevails, a marked lowering of the general standard is an unavoidable consequence.

The solution of triangles, given three sides, is by means of the formula $\tan \frac{1}{2}A = r/(s-a)$, as advocated by Mr. Hope-Jones in the *Gazette* of July 1935, and as used generally in France and probably other countries. This formula is undoubtedly better than any other half-angle formula. The formulae for $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ are not listed, and here, too, we should do well to copy. The author recommends the use of Mollweide's equation

$$(a+b)/c = \cos \tfrac{1}{2}(A-B)/\sin \tfrac{1}{2}C$$

as a general check in the solution of triangles as it uses all the sides and angles. It is unnecessary to go beyond the logarithms of both sides, so the check does not take long.

Cosecant is abbreviated to csc, which appears to be general in America as well as on the Continent. This is obviously more convenient than cosec, and we should adopt the abbreviation here. But why not go further? Could we not use s, c, t, ct and sc instead of sin, cos, tan, cot and sec? S. I.

Rinehart's Mathematical Tables. Compiled by H. D. LARSEN. Pp. 264. 15s. 1949. (Chapman and Hall)

The tables in this book occupy 160 pages. There follows a list of formulae (36 pages), graphs of well-known functional relations (14 pages), a list of standard forms in calculus (41 pages), and an additional 5 pages on elementary series. In the integration section 430 standard forms are listed; the author seems to have left nothing to the possible knowledge or skill of the reader. We have the integrals of $(ax+b)^n$, $1/(ax+b)$, $1/(ax+b)^2$, $x/(ax+b)^2$, $x/(ax+b)^3$, $x^2(ax+b)^n$, $x^2/(ax+b)$, $x^2/(ax+b)^2$, $x^2/(ax+b)^3$; but why stop there? As this scheme is used generally, it is easy to see how 430 "standard forms" are obtained.

The logarithm tables are to 5 decimal places and occupy 18 pages. Two pages are used for numbers 100–200, two for 200–300, and so on, but instead of these pages facing each other, they are on different sides of the same sheet. The differences for the 5th figures are obtained by proportional parts 0·1, 0·2, ... , 0·9 of various differences in the manner of the well-known Chambers'

Tables. The proportional parts are given to the 6th decimal place. Chambers' 7-figure logarithms occupy 180 pages. In 18–20 pages it should be possible to give 6-figure logarithms, so the compiler has not made good use of his space. The trigonometrical functions and their logarithms are tabulated at intervals of 1′. These occupy 74 pages and seem to call for 6 rather than 5 decimal places. The secants, cosecants and their logarithms are not given. Values of N^2, $\sqrt{N}$, $\sqrt{(10N)}$, N^3, $\sqrt[3]{(10N)}$, $\sqrt[3]{(100N)}$ and $1000/N$ are given for

$$1 \leqslant N \leqslant 1000.$$

There is a certain amount of redundancy. For example, we have (Table 5) 4-figure logarithms (no fourth-figure differences) and 5 place tables of the circular functions and their logarithms at intervals of 0·1° (Tables 6, 7). There are degrees in terms of radians and *vice versa*, the logarithms of circular functions when the angle is in radians and the logarithms of factorials of the numbers 1–250. We have the values of $\log_e x \, (1 \leqslant x \leqslant 100)$, e^x, $\log_{10} e^x$, e^{-x}, and the hyperbolic functions, all for x between 0 and 10. There are compound interest and mortality tables and useful sets of tables required in Statistics. The printing and spacing are excellent and the tables are easy to read. The 100 pages which follow the tables might well have been omitted. S. I.

Jacob Steiner's Geometrical Constructions with a Ruler. Translated by M. E. Stark, edited by R. C. Archibald. Pp. 88. $2. 1950. (Scripta Mathematica, Yeshiva University, New York)

Steiner's famous book on *Geometrical Constructions* was first published 117 years ago, and since then translations have appeared in French, Russian and Polish. The book under review is the first English translation and is number four in the Scripta Mathematica Studies Series.

Jacob Steiner, whose mathematical activity was entirely in the field of synthetic geometry, shows in this book how it is possible to solve " all " geometrical problems (in the plane) by means of the ruler alone, if any fixed circle in the plane is given. The solutions depend on eight basic constructions, which follow from certain properties of rectilinear figures and circles. In an Appendix twenty-two problems are solved, and of these twenty-one are projective and concern conics, *e.g.* if a conic is defined by three points and two tangents, find the points of contact of the tangents.

The book should be of special interest to teachers, and it will also appeal to those interested in the history of mathematics in the nineteenth century.

L. S. G.

Electromagnetic Theory. Pp. iv, 91. $3. 1950. Proceedings of Symposia in Applied Mathematics, 2. (American Mathematical Society)

This book represents the proceedings of a symposium in Electromagnetic Theory held at the M.I.T. in the summer of 1948. But in its present form it is exceedingly uneven. Thus there are four fairly substantial articles printed in full, and about a dozen short abstracts ranging in length from eleven lines to a couple of pages each. Since these latter vary from a study of stochastic (random) processes and information theory to a new theory of radiation and the propagation of centimetre electric waves in horns, there is no coherence, and anyone who attempts to read them all will soon find the meal much too varied to digest. Indeed, it is very hard to see why the volume was produced in its present form. Of the two longest papers, one of twenty pages by H. Feshbach gives an interesting general account of the new quantum electrodynamics in which the usual infinities are avoided, and the other of twenty-eight pages by J. L. Synge introduces an electromagnetism without metric. In addition to these there is a short summary (eight pages) by L. Infeld of

the powerful factorisation method of dealing with many of the simpler eigenvalue problems in physics and an amusing attempt by W. H. Watson to introduce discontinuous processes of creation and annihilation of charge into Maxwell's equations. To a person who was present at the Symposium, this book may serve as a reminder of what transpired. But other people will be tempted to protest that such a mixed bag is rather dear. C. A. COULSON.

Les Principes de l'Analyse Géométrique. Tome II. Fasc. (A). Base Méthodologique. By G. BOULIGAND. Pp. xxi, 209. 1100 fr. 1950. (Vuibert, Paris)

This highly readable work, dedicated to Professor M. Fréchet on his scientific jubilee, is the first part of the second volume of a large treatise. The first volume has already been reviewed in the *Gazette* by Dr. Whitrow, but although it is not infrequently referred to in the text, it is not essential to have read it before attempting this one. The book provides a most helpful introduction to the vocabulary and fundamental notions of modern abstract mathematics. Nothing seems to be lacking in accuracy or lucidity of presentation, and the author succeeds admirably in introducing an extensive array of abstract ideas, while all the time emphasising their natural origins and providing most illuminating comment on, and background to, the subject-matter. In dealing with the ideas with which the reviewer is most familiar, the author has usually managed to provide a stimulating and refreshing point of view, and on less familiar ground he is always a great source of illumination.

The scope of the work can be deduced from the subject-matter of the various chapters. Following a longish introduction discussing elementary questions in the theory of analytic functions, the first chapter is devoted to fundamental ideas, including the notions of equivalence relation and correspondence between sets, and includes also a discussion of the axiom of choice. Chapters II and III are devoted to groups and Chapter IV discusses rings, ideals and the notion of an algebraic variety. Chapter V deals with fundamental topological notions, and Chapter VI with combinatorial topology, measure theory and abstract integration, and lattice theory (a mixed grill which is surprisingly easy to digest). The next two chapters introduce more advanced topological ideas : continuity and ideas deriving from it are discussed in Chapter VII, and connectedness and related notions in the far-reaching Chapter VIII. Chapter IX begins with an abstract discussion of the convergence of sequences and proceeds to get to work on a number of interesting and sensible problems. It includes some discussion on ordinary differential equations, and on the general notion of successive approximations with applications to integral equations. A final chapter entitled " Vues récapitulaires et compléments divers " rounds the book off in a thoroughly edifying manner.

This book can be recommended to widely different classes of readers. Firstly to the mathematician who thinks he knows, or at least knows that he ought to know, the greater part of the subject-matter discussed, the book will be a valuable work of reference (though not so valuable as it would be if indexed) and a most useful source of expository ideas. To the would-be research student who is anxious to acquire a firm grasp of the abstract ideas discussed, this book provides a fairly painless route to knowledge. Every mathematical graduate should find something that takes his fancy : there are new lights on what he has already learnt, exciting and understandable generalisations of familiar notions, and an introduction provided to further reading on any subject which sufficiently arouses his interest. Finally, anyone who has been interested by and enjoyed reading the masterly work of Courant and Robbins, and is looking for fresh fields to conquer, should find something to his taste here. D. B. S.

An Introduction to the Theory of Equations. By RAM BEHARI and HANSRAJ GUPTA. Pp. 169. Rs. 7/8. 1947. (Chand, Delhi)

As stated in the preface : " This book aims at providing elementary knowledge of the theory of equations and is planned to cover the syllabuses of the B.A. and B.Sc. (Pass and Honours) and M.A. examinations of Indian universities."

Emphasis is laid on the more practical aspects of the subject, and there are plenty of examples, many of which are actually solved in the text. Unfortunately the typography is not good, and there are several printing errors.

P. M. H.

A Course of Mathematical Analysis. By SHANBI NARAYAN. Second edition. Pp. iv, 304. Rs. 15. 1949. (Chand, Delhi)

This book is intended for students who already have some knowledge of the calculus, based on the geometrical and intuitively perceived notion of the continuum, and now wish to proceed to a serious study of the purely arithmetical theory of functions of a real variable. The treatment of the subject is essentially rigorous, and apart from the poor printing the presentation is clear and lucid.

Commencing with the elements of the theory of aggregates of real numbers the author leads us to the consideration of sequences and infinite series. He then introduces the idea of a function and defines differentiability and continuity before discussing the Riemann theory of the definite integral. This is dealt with in great detail and is, in the opinion of the reviewer, the best part of the book. The concept of uniform convergence is used to deal with the analytic theory of trigonometrical functions ; functions of several variables are studied and applied in considering definite integrals as functions of a parameter. The book ends with a brief discussion of Fourier series. P. M. H

The Theory of Group Characters and the Matrix Representation of Groups. By D. E. LITTLEWOOD. Second edition. Pp. viii, 310. 25s. 1950. (Geoffrey Cumberlege, Oxford University Press)

Professor Littlewood's book on its appearance was recognised as the work of an expert algebraist, though not, perhaps, too easy reading for the novice, in spite of the fact that the earlier chapters provide a self-contained account of matrices, algebras and groups. The new edition is substantially a reprint, but there is an addition of some sixteen pages to the appendix, describing recent developments, chiefly connected with invariant theory and tensors ; much of this work is the author's own, and is to be found in detail in his papers published during the last ten years in *Phil. Trans. Roy. Soc.*, *Proc. London Math. Soc.*, and *Proc. Cambridge Phil. Soc.*

Elementary Analytical Conics. By J. H. SHACKLETON BAILEY. Second edition. Pp. 378. 8s. 6d. 1950. (Geoffrey Cumberlege, Oxford University Press)

Dr. Shackleton Bailey's book, now appearing in a second edition, is written specifically for Higher Certificate candidates, and contains an admirable selection of exercises set in Higher Certificate examinations. Those teachers who feel that coordinate geometry should be made to flow naturally from two or three general principles will hardly welcome a book which tends to exalt *ad hoc* devices ; but those whose task is to produce successful candidates rather than sound geometers may find the volume helpful. T. A. A. B.

FILM STRIPS.

Treasures of the Triangle. By G. H. Grattan-Guinness.

Part 1. Concurrent Lines (*a*) O and I. 37 frames. Teaching notes. Pp. 8.

Part 2. Concurrent Lines (*b*) G and H. 36 frames. Teaching notes. Pp. 8.

Part 3. Collinear Points. 42 frames. Teaching notes. Pp. 8.

1950. (Educational Productions Ltd.)

The tradition of the early days of the Mathematical Association, when the improvement of the teaching of geometry occupied all its thoughts, will make its members particularly interested in the introduction of a very modern teaching aid, the film strip, into the same field. In a sequence of excellent diagrams, the author has set out thematically the concurrence properties of circumcentre, incentre and excentres, centroid and orthocentre; the collinearity of points of intersection of angle bisectors with opposite sides, and the Simson line. There is, of course, no attempt at proof; this would either follow or precede, according as the film strip is used to awaken interest in a new topic, or to bind together the results of earlier work.

This is within the normal scope of the Grammar School and has its place in the middle school or in the sixth. The author suggests other possibilities; that within the Modern School, where formal proof would be out of reach, these film strips may be a means of introducing some new geometrical ideas, as a source of interest, if not of amazement; hence the title. It is doubtful whether there exists, so far, sufficient experience of this type of treatment upon which to base a sound opinion of its validity. Certainly this courageous experiment is worth trying, if only to relieve the tedium of the endless drawing which goes under the name of geometry in many such schools.

To this end the theme has been developed very carefully. The choice of diagram and the technique of production are excellent. The accuracy of drawing and the contrast of shades produce a quality of diagram which is out of reach of most teachers within the limitations of the blackboard. The notes which accompany the strips set out this and other points in the case for their use in teaching, together with a description of each frame. The teacher who is interested in modern methods will spare a few moments from the eternal round of problem-solving to show these strips to his pupils.

I. R. V.

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HENRY RONALD HASSÉ, D.Sc.

President, January 1950—March 1951.